This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

EducT
148
75.435

Harvard College
Library

VE RI
TAS

BOUGHT
FROM THE GIFT OF
CHARLES HERBERT THURBER

3 2044 097 014 401

Educ T 148.75.435

NEW

Elementary Geometry,

WITH

PRACTICAL APPLICATIONS.

A SHORTER COURSE,

UPON THE BASIS OF THE LARGER WORK:

BY

BENJAMIN GREENLEAF, A.M.,

AUTHOR OF A MATHEMATICAL SERIES.

BOSTON:
PUBLISHED BY ROBERT S. DAVIS & CO.
NEW YORK: BAKER, PRATT, & CO., 142 & 144 GRAND STREET.
PHILADELPHIA: J. A. BANCROFT & COMPANY.
ST. LOUIS: GRAY, BAKER, & COMPANY.
CHICAGO: JANSEN, MCCLURG, & CO.
1875.

Educ T 149.73.435
HARVARD COLLEGE LIBRARY
FROM THE GIFT OF
CHARLES HERBERT THURBER
Mar. 5, 1926

GREENLEAF'S

New Comprehensive Series;

AN ENTIRELY NEW MATHEMATICAL COURSE,

Fully adapted to the best methods of modern instruction.

New Primary Arithmetic.
New Elementary Arithmetic.
New Intellectual Arithmetic.
New Practical Arithmetic.
New Elementary Algebra.
New Elementary Geometry.

New Higher Algebra.
Elements of Geometry.
Elements of Trigonometry.
Geometry and Trigonometry, in one volume.

Common School Arithmetic.
National Arithmetic.

Entered according to Act of Congress, in the year 1873, by
Robert S. Davis & Company,
In the office of the Librarian of Congress, at Washington.

PREFACE.

THE amount of time usually devoted to the study of Geometry in High Schools and Academies is not sufficient for the mastery of a lengthy treatise. The present edition of Greenleaf's Elements of Geometry has been prepared to meet the wants of such institutions. It consists of all the more important theorems and problems of the science, and contains, therefore, all the Geometry required for admission into the best Scientific Schools and Colleges of the country.

The arrangement and language of the "Elements" having been commended by the best mathematical teachers, no change has been made in either, except as appeared absolutely necessary to adapt the same to the proper character of a briefer and more elementary work. This book is, then not wholly a new work; but a more compendious edition of a standard manual that has proved its excellence by standing the test of the school-room.

The exercises which have been freely introduced are intended to test the thoroughness of the learner's geometrical knowledge, besides being especially adapted to develop skill and discrimination in the demonstration of theorems, and in the solution of problems, unaided except by definitions and principles.

The Practical Applications, or Exercises in Mensuration, given at the close of the volume, can be used at the discretion of the teacher, after the completion of Book III., in connection with the text, to which they refer, as the pupil progresses.

The course can be limited wholly to Plane Geometry by omitting all after Book IV.

In general, Geometry follows Algebra in a full Mathematical Course; but in a limited course, the first four books of this Manual may often be read with advantage by the Student, even before entering upon Higher Arithmetic, or Elementary Algebra.

The larger work, known as Greenleaf's Elements of Geometry, especially adapted to the wants of Higher Seminaries, and now extensively used in them, will continue to be published.

It is proper to state that this volume has been prepared by H. B. Maglathlin, whose valuable labors have been before acknowledged in several books of the series.

Boston, May, 1873.

CONTENTS.

BOOK I.

PAGE.
GENERAL PRINCIPLES 7
Preliminary Definitions 7
Theorems 13
Exercises for original thought, on review . . . 33

BOOK II.

RATIO AND PROPORTION 34
Definitions 34
Theorems 37
Exercises for original thought, on review . . . 42

BOOK III.

THE CIRCLE AND THE MEASURE OF ANGLES . . . 44
Definitions 44
Theorems 46
Problems in Construction 59
Exercises for original thought, on review . . . 66

BOOK IV.

AREAS AND RELATIONS OF PLANE FIGURES 68
Definitions 68
Theorems 69
Problems in Construction 104
Exercises for original thought, on review . . . 108

BOOK V.

Planes and Polyedrons 110
Definitions 110
Theorems 114
Exercises for original thought, on review . . . 138

BOOK VI.

The Three Round Bodies 139
Definitions 139
Theorems 142
Exercises for original thought, on review . . . 155

Practical Applications 156
Mensuration of Lines and Surfaces 156
Mensuration of Volumes 167
Miscellaneous Exercises 175

FOR TEACHERS.

A Key to New Elementary Geometry has been prepared, and published for the convenience of teachers, which will be mailed prepaid, on receipt of 50 cents by the Publishers.

ELEMENTARY GEOMETRY.

BOOK I.

GENERAL PRINCIPLES.

PRELIMINARY DEFINITIONS.

1. **Geometry** is the science of position and extension.

The elements of position are direction and distance.

The dimensions of extension are length, breadth, and thickness.

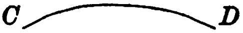
2. A **Point** is that which has position, without magnitude.

3. A **Line** is that which has length, without either breadth or thickness.

4. A **Straight Line** is one which has the same direction in its whole extent; as the line *A B*.

The word *line* is frequently used to designate a straight line.

5. A **Curved Line** is one which continually changes its direction; as the line *C D*.



The word *curve* is frequently used to designate a curved line.

6. A **Surface** is that which has length and breadth, without hight or thickness.

7. A **Plane Surface**, or simply a PLANE, is one in which any two points being taken, the straight line that joins them will lie wholly in the surface.

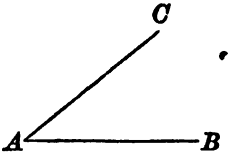
8. A **Curved Surface** is one that is not a plane surface, nor made up of plane surfaces.

9. A **Solid**, or **Volume**, is that which has length, breadth, and thickness.

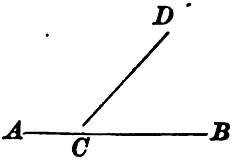
ANGLES AND LINES.

10. An **Angle** is the difference in the direction of two lines, which meet at a point; as the angle *A*.

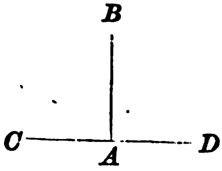
The point of meeting, *A*, is the *vertex* of the angle, and the lines *A B*, *A C* are the *sides* of the angle.



An angle may be designated, not only by the letter at its vertex, as *C*, but when two or more angles have the same vertex, the letter at the vertex always occupies the middle place; as the angle *A C D* or *D C B*.

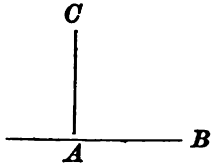


11. Two straight lines are said to be **Perpendicular** to each other, when their meeting forms equal adjacent angles; thus the lines *A B* and *C D* are perpendicular to each other.

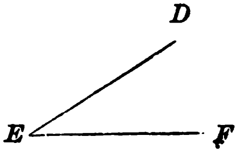


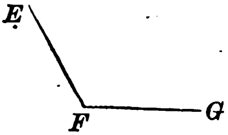
Two adjacent angles, as *C A B* and *B A D*, have a common vertex, as *A*; and a common side, as *A B*.

12. A **Right Angle** is one which is formed by a straight line and a perpendicular to it; as the angle *C A B*.



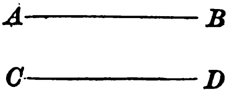
13. An **Acute Angle** is one which is less than a right angle; as the angle *D E F*.



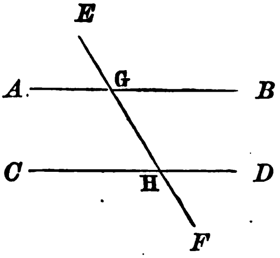


An Obtuse Angle is one which is greater than a right angle; as the angle *E F G*.

Acute and obtuse angles are sometimes called *oblique angles.*



14. Parallel Lines are such as, being in the same plane, cannot meet, however far either way both of them may be produced; as the lines *A B*, *C D*.



15. When a straight line, as *E F*, intersects two parallel lines, as *A B*, *C D*, the angles formed by the intersecting or secant line take particular names, thus: —

Interior Angles on the same Side are those which lie within the parallels, and on the same side of the secant line; as the angles *B G H*, *G H D*, and also *A G H*, *G H C*.

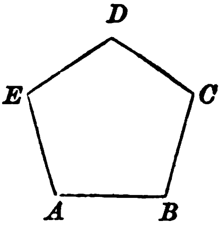
Alternate Interior Angles lie within the parallels, and on different sides of the secant line, but are not adjacent to each other; as the angles *B G H*, *G H C*, and also *A G H*, *G H D*.

Opposite Exterior and Interior Angles lie on the same side of the secant line, the one without and the other within the parallels, but not adjacent to each other; as the angles *E G B*, *G H D*, and also *E G A*, *G H C*, are, respectively, the opposite exterior and interior angles.

PLANE FIGURES.

16. A Plane Figure is a plane terminated on all sides by straight lines or curves.

The boundary of any figure is called its *perimeter*.

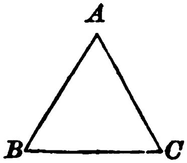


17. A **Polygon** is a plane bounded by straight lines; as the figure *A B C D E.*

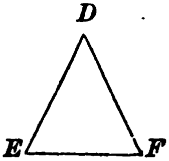
A polygon of three sides is called a *triangle;* one of four sides, a *quadrilateral;* one of five, a *pentagon;* one of six, a *hexagon;* one of seven, a *heptagon;* one of eight, an *octagon;* one of nine, a *nonagon;* one of ten, a *decagon;* one of eleven, an *undecagon;* one of twelve, a *dodecagon;* and so on.

A polygon which has equal sides and equal angles is called a *regular polygon;* and the equal sides and equal angles are called *homologous* sides and angles.

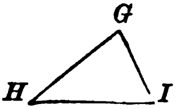
18. An **Equilateral Triangle** is one which has its three sides equal; as the triangle *A B C.*



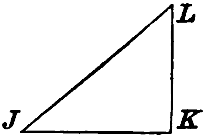
An **Isosceles Triangle** is one which has two of its sides equal; as the triangle *D E F.*



A **Scalene Triangle** is one which has no two of its sides equal; as the triangle *G H I.*



19. A **Right-angled Triangle** is one which has a right angle; as the triangle *J K L.*

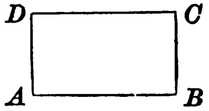


The side opposite to the right angle is called the *hypothenuse;* as the side *J L.*

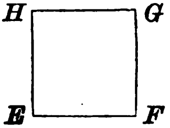
An *Acute-angled Triangle* is one which has three acute angles. An *Obtuse-angled Triangle* is one which has an obtuse angle. Acute-angled and obtuse-angled triangles are also called *Oblique-angled* triangles.

20. A **Parallelogram** is a quadrilateral which has its opposite sides parallel.

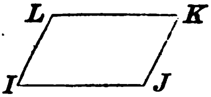
21. A **Rectangle** is any parallelogram whose angles are right angles; as the parallelogram *A B C D*.



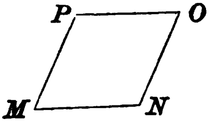
A **Square** is a rectangle whose sides are equal; as the rectangle *E F G H*.



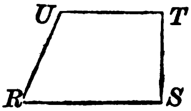
22. A **Rhomboid** is any parallelogram whose angles are not right angles; as the parallelogram *I J K L*.



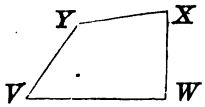
A **Rhombus** is a rhomboid whose sides are equal; as the rhomboid *M N O P*.



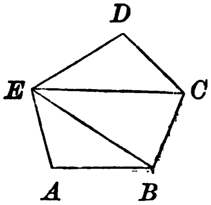
23. A **Trapezoid** is a quadrilateral which has only two of its sides parallel; as the quadrilateral *R S T U*.



A **Trapezium** is a quadrilateral which has no two of its sides parallel; as the quadrilateral *V W X Y*.



24. A **Diagonal** is a line joining the vertices of any two angles which are opposite to each other; as the lines *E C* and *E B* in the polygon *A B C D E*.



25. A **Base** of a polygon is the side on which the polygon is supposed to stand. But in the case of the isosceles triangle, it is usual to consider that side the base which is not equal to either of the other sides.

AXIOMS.

26. An **Axiom** is a self-evident truth; such as, —

1. Things which are equal to the same thing, are equal to each other.
2. If equals be added to equals, the sums will be equal.
3. If equals be taken from equals, the remainders will be equal.
4. If equals be added to unequals, the sums will be unequal.
5. If equals be taken from unequals, the remainders will be unequal.
6. Things which are double of the same thing, or of equal things, are equal to each other.
7. Things which are halves of the same thing, or of equal things, are equal to each other.
8. The whole is greater than any of its parts.
9. The whole is equal to the sum of all its parts.
10. A straight line is the shortest line that can be drawn from one point to another.
11. All right angles are equal to one another.
12. Magnitudes which coincide throughout their whole extent, are equal.

POSTULATES.

27. A **Postulate** is a self-evident problem; such as, —

1. That a straight line may be drawn from one point to another.
2. That a straight line may be produced to any length.
3. That a straight line may be drawn through a given point parallel to another straight line.
4. That an angle may be described equal to any given angle.

PROPOSITIONS.

28. A **Theorem** is something to be demonstrated.

29. A **Problem** is something to be performed.

30. A **Demonstration** is a course of reasoning by which a truth becomes evident.

31. A **Proposition** is something proposed to be demonstrated, or to be performed.

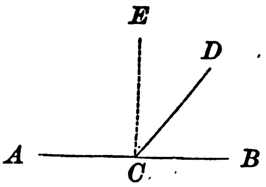
32. A **Corollary** is an obvious consequence deduced from one or more propositions.

33. A **Scholium** is a remark made upon one or more preceding propositions.

34. An **Hypothesis** is a supposition, made either in the enunciation of a proposition, or in the course of a demonstration.

THEOREM I.

35. *The adjacent angles which one straight line makes by meeting another straight line, are together equal to two right angles.*

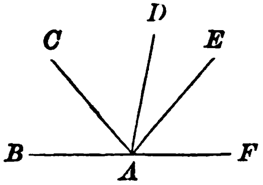


Let the straight line $D\,C$ meet $A\,B$, making the adjacent angles $A\,C\,D$, $D\,C\,B$; these angles together will be equal to two right angles.

From the point C suppose $C\,E$ to be drawn perpendicular to $A\,B$; then the angles $A\,C\,E$ and $E\,C\,B$ will each be a right angle (12). The angle $A\,C\,D$ is composed of the right angle $A\,C\,E$ and the angle $E\,C\,D$ (26, Ax. 9); but this angle $E\,C\,D$ and $D\,C\,B$ compose the other right angle, $E\,C\,B$; hence the angles $A\,C\,D$, $D\,C\,B$ together equal two right angles.

36. *Cor.* 1. If one of the angles $A\,C\,D$, $D\,C\,B$ is a right angle, the other must also be a right angle.

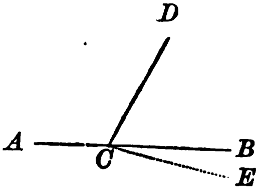
37. *Cor.* 2. All the successive angles, in the same plane, $B\,A\,C$, $C\,A\,D$, $D\,A\,E$, $E\,A\,F$, formed at a single point, on the same side of a straight line, $B\,F$, are equal, when taken together, to two right angles; for their sum is equal to that of the two adjacent angles, $B\,A\,C$, $C\,A\,F$.



THEOREM II.

38. *If one straight line meets two other straight lines at a common point, making adjacent angles, which together are equal to two right angles, the two lines form one and the same straight line.*

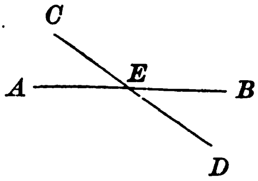
Let the straight line $D\,C$ meet the two straight lines $A\,C$, $C\,B$ at the common point C, making the adjacent angles $A\,C\,D$, $D\,C\,B$ together equal to two right angles; then the lines $A\,C$ and $C\,B$ will form one and the same straight line.



If $C\,B$ is not the straight line $A\,C$ produced, let $C\,E$ be that line produced; then the line $A\,C\,E$ being straight, the sum of the angles $A\,C\,D$ and $D\,C\,E$ will be equal to two right angles (Theo. I.). But by hypothesis the angles $A\,C\,D$ and $D\,C\,B$ are together equal to two right angles; therefore the sum of the angles $A\,C\,D$ and $D\,C\,E$ must be equal to the sum of the angles $A\,C\,D$ and $D\,C\,B$ (26, Ax. 2). Take away the common angle $A\,C\,D$ from each, and there will remain the angle $D\,C\,B$, equal to the angle $D\,C\,E$, a part to the whole, which is impossible; therefore $C\,E$ is not the line $A\,C$ produced. Hence $A\,C$ and $C\,B$ form one and the same straight line.

THEOREM III.

39. *When two straight lines intersect each other, the opposite or vertical angles which they form are equal.*



Let the two straight lines $A B$, $C D$ intersect each other at the point E; then will the angle $A E C$ be equal to the angle $D E B$, and tne angle $C E B$ to $A E D$.

For the angles $A E C$, $C E B$, which the straight line $C E$ forms by meeting the straight line $A B$, are together equal to two right angles (Theo. I.); and the angles $C E B$, $B E D$, which the straight line $B E$ forms by meeting the straight line $C D$, are equal to two right angles; hence the sum of the angles $A E C$, $C E B$ is equal to the sum of the angles $C E B$, $B E D$ (26, Ax. 1). Take away from each of these sums the common angle $C E B$, and there will remain the angle $A E C$, equal to its opposite angle, $B E D$ (26, Ax. 3).

In the same manner it may be shown that the angle $C E B$ is equal to its opposite angle, $A E D$.

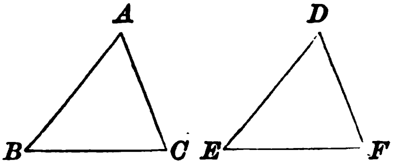
40. *Cor.* 1. The four angles formed by two straight lines intersecting each other, are together equal to four right angles.

41. *Cor.* 2. All the successive angles, in the same plane, around a common point, formed by any number of straight lines meeting at that point, are together equal to four right angles.

THEOREM IV.

42. *If two triangles have two sides and the included angle in the one equal to two sides and the included angle in the other, each to each, the two triangles will be equal.*

In the two triangles $A B C$, $D E F$, let the side $A B$ be



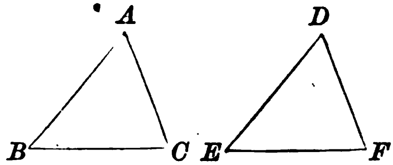
equal to the side DE, the side AC to the side DF, and the angle A to the angle D; then the triangles ABC, DEF will be equal.

Conceive the triangle ABC to be applied to the triangle DEF, so that the side AB shall fall upon its equal, DE, the point A upon D, and the point B upon E; then, since the angle A is equal to the angle D, the side AC will take the direction DF. But AC is equal to DF; therefore the point C will fall upon F, and the third side BC will coincide with the third side EF (26, Ax. 10). Hence the triangle ABC coincides with triangle DEF, and they are therefore equal (26, Ax. 12).

43. *Cor.* When, in two triangles, these three parts are equal, namely, the side AB equal to DE, the side AC equal to DF, and the angle A equal to D, the other three corresponding parts are also equal, namely, the side BC equal to EF, the angle B equal to E, and the angle C equal to F.

THEOREM V.

44. *If two triangles have two angles and the included side in the one equal to two angles and the included side in the other, each to each, the two triangles will be equal.*



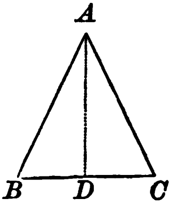
In the two triangles ABC, DEF, let the angle B be equal to the angle E, the angle C to the angle F, and the side BC to the side EF; then the triangles ABC, DEF will be equal.

Conceive the triangle ABC to be applied to the triangle DEF, so that the side BC shall fall upon its equal, EF, the point B upon E, and the point C upon F. Then, since the angle *B* is equal to the angle *E*, the side *B A* will take the direction *E D;* therefore the point *A* will be found somewhere in the line *E D*. In like manner, since the angle *C* is equal to the angle *F*, the line *C A* will take the direction *F D*, and the point *A* will be found somewhere in the line *F D*. Hence the point *A*, falling at the same time in both of the straight lines *E D* and *F D*, must fall at their intersection *D*. Hence the two triangles *A B C*, *D E F* coincide with each other, and are therefore equal (26, Ax. 12).

45. *Cor.* When, in two triangles, these three parts are equal, namely, the angle *B* equal to the angle *E*, the angle *C* equal to the angle *F*, and the side *B C* equal to the side *E F*, the other three corresponding parts are also equal; namely, the side *B A* equal to *E D*, the side *C A* equal to *F D*, and the angle *A* equal to the angle *D*.

THEOREM VI.

46. *In an isosceles triangle, the angles opposite the equal sides are equal.*

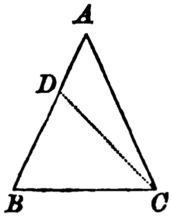


Let *A B C* be an isosceles triangle, in which the side *A B* is equal to the side *A C;* then will the angle *B* be equal to the angle *C*.

Conceive the angle *B A C* to be bisected, or divided into two equal parts, by the straight line *A D*, making the angle *B A D* equal to *D A C*. Then the two triangles *B A D*, *C A D* have the two sides *A B*, *A D* and the included angle in the one equal to the two sides *A C*, *A D* and the included angle in the other, each to each; hence the two triangles are equal, and the angle *B* is equal to the angle *C* (Theo. IV.).

THEOREM VII.

47. *If two angles of a triangle are equal, the opposite sides are also equal, and the triangle is isosceles.*

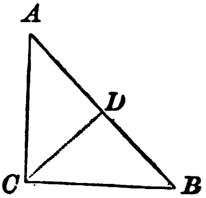


Let $A\,B\,C$ be a triangle having the angle B equal to the angle C; then will the side $A\,B$ be equal to the side $A\,C$.

For, if the two sides are not equal, one of them must be greater than the other. Let $A\,B$ be the greater; then take $D\,B$ equal to $A\,C$ the less, and draw $C\,D$. Now, in the two triangles $D\,B\,C$, $A\,B\,C$, we have $D\,B$ equal to $A\,C$ by construction, the side $B\,C$ common, and the angle B equal to the angle $A\,C\,B$ by hypothesis; therefore, since two sides and the included angle in the one are equal to two sides and the included angle in the other, each to each, the triangle $D\,B\,C$ is equal to the triangle $A\,B\,C$ (Theo. IV.), a part equal to the whole, which is impossible (26, Ax. 8). Hence the sides $A\,B$ and $A\,C$ cannot be unequal; therefore the triangle $A\,B\,C$ is isosceles.

THEOREM VIII.

48. *The greater side of any triangle is opposite the greater angle.*



In the triangle $C\,A\,B$, let the angle C be greater than B; then will the side $A\,B$, opposite to C, be greater than $A\,C$, opposite to B.

Draw the straight line $C\,D$, making the angle $B\,C\,D$ equal to B. Then, in the triangle $B\,D\,C$, we shall have the side $B\,D$ equal to $D\,C$ (Theo. VII.). To each add $D\,A$, and we have $B\,D$ plus $D\,A$ equal to $D\,C$ plus $D\,A$; but $D\,C$ plus $D\,A$ is greater than $A\,C$ (26, Ax. 10). Hence $B\,D$ plus $D\,A$, or $A\,B$, is greater than $A\,C$.

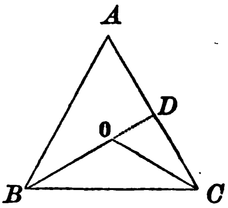
49. *Cor.* 1. Therefore the shorter side is opposite to the less angle.

50. *Cor.* 2. The greater angle is opposite to the greater side. For, if the angle C is not greater than B, it must be equal to it or less. If the angle C were equal to B, then would the side $A\,B$ be equal to the side $A\,C$ (Theo. VII.), which is contrary to the hypothesis; and if the angle C were less than B, then would the side $A\,B$ be less than $A\,C$, which is also contrary to the hypothesis. Hence, the angle C must be greater than B.

51. *Cor.* 3. **It follows, therefore, that** the less angle is opposite to the shorter side.

THEOREM IX.

52. *If, from any point within a triangle, two straight lines are drawn to the extremities of either side, their sum will be less than that of the other two sides of the triangle.*



Let the two straight lines $B\,O$, $C\,O$ be drawn from the point O, within the triangle $A\,B\,C$, to the extremities of the side $B\,C$; then will the sum of the two lines $B\,O$ and $O\,C$ be less than the sum of the sides $B\,A$ and $A\,C$.

Let the straight line $B\,O$ be produced till it meets the side $A\,C$ in the point D; and because one side of a triangle is less than the sum of the other two sides (26, Ax. 10), the side $O\,C$ in the triangle $C\,D\,O$ is less than the sum of $O\,D$ and $D\,C$. To each of these inequalities add $B\,O$, and we have the sum of $B\,O$ and $O\,C$ less than the sum of $B\,O$, $O\,D$, and $D\,C$ (26, Ax. 4); or the sum of $B\,O$ and $O\,C$ less than the sum of $B\,D$ and $D\,C$. Again, because the side $B\,D$ is less than the sum of $B\,A$ and $A\,D$, by adding $D\,C$ to each, we have the sum of $B\,D$ and $D\,C$ less than the sum of $B\,A$ and $A\,C$. But it has been just shown that the sum of $B\,O$ and $O\,C$ is less than the sum of $B\,D$ and $D\,C$; much more, then, is the sum of $B\,O$ and $O\,C$ less than $B\,A$ and $A\,C$.

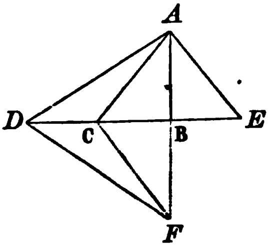
THEOREM X.

53. *If, from a point without a straight line, a perpendicular be let fall on that line, and oblique lines be drawn to different points in the same line;—*

1st. *The perpendicular will be shorter than any oblique line.*

2d. *Any two oblique lines, which meet the given line at equal distances from the perpendicular, will be equal.*

3d. *Of any two oblique lines, that which meets the given line at the greater distance from the perpendicular will be the longer.*



Let A be the given point, and $D\,E$ the given straight line. Draw $A\,B$ perpendicular to $D\,E$, and the oblique lines $A\,E$, $A\,C$, $A\,D$. Produce $A\,B$ till $B\,F$ is equal to $A\,B$, and join $C\,F$, $D\,F$.

First. The triangle $B\,C\,F$ is equal to the triangle $B\,C\,A$, for they have the side $C\,B$ common, the side $A\,B$ equal to the side $B\,F$, and the angle $A\,B\,C$ equal to the angle $F\,B\,C$, both being right angles (Theo. I. Cor. 1); hence the third sides, $C\,F$ and $A\,C$, are equal (Theo. V. Cor.). But $A\,B\,F$, being a straight line, is shorter than $A\,C\,F$, which is a broken line (26, Ax. 10); therefore $A\,B$, the half of $A\,B\,F$, is shorter than $A\,C$, the half of $A\,C\,F$; hence the perpendicular is shorter than any oblique line.

Secondly. If $B\,E$ is equal to $B\,C$, then, since $A\,B$ is common to the triangles $A\,B\,E$, $A\,B\,C$, and the angles $A\,B\,E$, $A\,B\,C$ are right angles, the two triangles are equal (Theo. IV.), and the side $A\,E$ is equal to the side $A\,C$ (Theo. IV. Cor.). Hence the two oblique lines, meeting the given line at equal distances from the perpendicular, are equal.

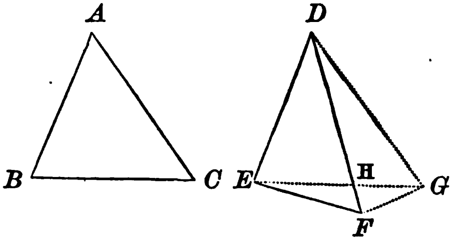
Thirdly. The point C being in the triangle $A\,D\,F$, the sum of the lines $A\,C$, $C\,F$ is less than the sum of the sides $A\,D$, $D\,F$ (26, Ax. 10). But $A\,C$ has been shown to be equal to $C\,F$; and in like manner it may be shown that $A\,D$ is equal to $D\,F$. Therefore $A\,C$, the half of the line $A\,C\,F$, is shorter than $A\,D$, the half of the line $A\,D\,F$; hence the oblique line which meets the given line the greater distance from the perpendicular, is the longer.

54. *Cor.* 1. The perpendicular measures the shortest distance of any point from a straight line; and, also, from the same point to a given straight line only two equal straight lines can be drawn.

55. *Cor.* 2. If a straight line have two points equally distant from the extremities of another straight line, it will be perpendicular to that line at its middle part.

THEOREM XI.

56. *If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angle of the one greater than the included angle of the other, the third side of that which has the greater angle will be greater than the third side of the other.*



Let $A\,B\,C$, $D\,E\,F$ be two triangles, having the side $A\,B$ equal to $D\,E$, and $A\,C$ equal to $D\,F$, and the angle A greater than D; then will the side $B\,C$ be greater than $E\,F$.

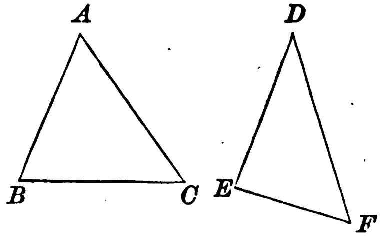
Of the two sides $D\,E$, $D\,F$, let $D\,F$ be the side which is not shorter than the other; make the angle $E\,D\,G$ equal to $B\,A\,C$; and make $D\,G$ equal to $A\,C$ or $D\,F$, and join $E\,G$, $G\,F$.

Since $D\,F$, or its equal $D\,G$, is not shorter than $D\,E$, it is longer than *D H* (Theo. X. Case 3); therefore its extremity, *F*, must fall below the line *E G*. The two triangles, *A B C* and *D E G*, have the two sides *A B*, *A C* equal to the two sides *D E*, *D G*, each to each, and the included angle *B A C* of the one equal to the included angle *E D G* of the other; hence the side *B C* is equal to *E G* (Theo. IV. Cor.).

In the triangle *D F G*, since *D G* is equal to *D F*, the angle *D F G* is equal to the angle *D G F* (Theo. VI.); but the angle *D G F* is greater than the angle *E G F*; therefore the angle *D F G* is greater than *E G F*, and much more is the angle *E F G* greater than the angle *E G F*. Because the angle *E F G* in the triangle *E F G* is greater than *E G F*, and because the greater side is opposite the greater angle (Theo. VIII.), the side *E G* is greater than *E F*; and *E G* has been shown to be equal to *B C*; hence *B C* is greater than *E F*.

THEOREM XII.

57. *If two triangles have two sides of the one equal to two sides of the other, each to each, but the third side of the one greater than the third side of the other, the angle contained by the sides of that which has the greater third side will be greater than the angle contained by the sides of the other.*

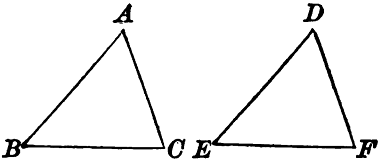


Let *A B C*, *D E F* be two triangles, the side *A B* equal to *D E*, and *A C* equal to *D F*, and the side *C B* greater than *E F*, then will the angle *A* be greater than *D*.

For, if it be not greater, it must either be equal to it or less. But the angle *A* cannot be equal to *D*, for then the side *B C* would be equal to *E F* (Theo. IV. Cor.), which is contrary to the hypothesis; neither can it be less, for then the side *B C* would be less than *E F* (Theo. XI.), which also is contrary to the hypothesis; therefore the angle *A* is not less than the angle *D*, and it has been shown that is not equal to it; hence the angle *A* must be greater than the angle *D*.

THEOREM XIII.

58. *If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles themselves will be equal.*



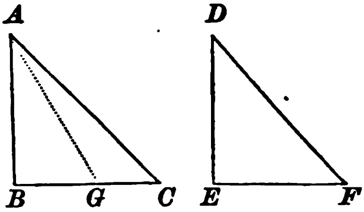
Let the triangles *A B C*, *D E F* have the side *A B* equal to *D E*, *A C* to *D F*, and *B C* to *E F;* then will the angle *A* be equal to *D*, the angle *B* to the angle *E*, and the angle *C* to the angle *F*, and the two triangles will also be equal.

For, if the angle *A* were greater than the angle *D*, since the sides *A B*, *A C* are equal to the sides *D E*, *D F*, each to each, the side *B C* would be greater than *E F* (Theo. XI.); and if the angle *A* were less than *D*, it would follow that the side *B C* would be less than *E F*. But by hypothesis *B C* is equal to *E F;* hence the angle *A* can neither be greater nor less than *D;* therefore it must be equal to it. In the same manner, it may be shown that the angle *B* is equal to *E*, and the angle *C* to *F;* hence the two triangles must be equal.

59. *Scholium.* In two triangles equal to each other, the equal angles are opposite the equal sides; thus the equal angles *A* and *D* are opposite the equal sides *B C* and *E F*.

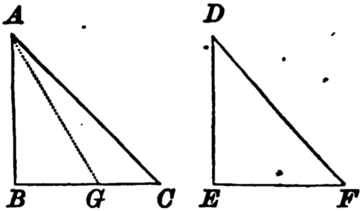
THEOREM XIV.

60. *If two right-angled triangles have the hypothenuse and a side of the one equal to the hypothenuse and a side of the other, each to each, the triangles are equal.*



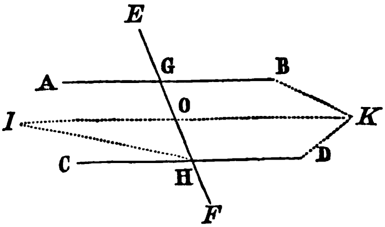
Let the two right-angled triangles ABC, DEF, have the hypothenuse AC equal to DF, and the side AB equal to DE; then will the triangle ABC be equal to the triangle DEF.

The two triangles are evidently equal, if the sides BC and EF are equal (Theo. XIII.). If it be possible, let these sides be unequal, and let BC be the greater. Take BG equal to EF, the less side, and join AG. Then, in the two triangles ABG, DEF, the angles B and E are equal, both being right angles, the side AB is equal to DE by hypothesis, and the side BG to EF by construction; hence these triangles are equal (Theo. IV.); and therefore AG is equal to DF. But by hypothesis DF is equal to AC, and therefore AG is equal to AC. But the oblique line AC cannot be equal to AG, which meets the same straight line nearer the perpendicular AB (Theo. X.); therefore BC and EF cannot be unequal, hence they must be equal; therefore the triangles ABC and DEF are equal.



THEOREM XV.

61. *If a straight line, intersecting two other straight lines, makes the alternate angles equal, the two lines are parallel.*

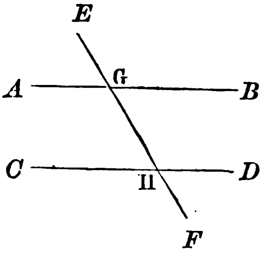


Let the straight line EF intersect the two straight lines AB, CD, making the alternate angles BGH, CHG equal; then the lines AB, CD will be parallel.

For, if the lines *A B*, *C D* are not parallel, let them meet in some point *K*, and through *O*, the middle point of *G H*, draw the straight line *I K*, making *I O* equal to *O K*, and join *H I*. Then the opposite angles *K O G*, *I O H*, formed by the intersection of the two straight lines *I K*, *G H*, are equal (Theo. III.); and the triangles *K O G*, *I O H* have the two sides *K O*, *O G* and the included angle in the one equal to the two sides *I O*, *O H* and the included angle in the other, each to each; hence the angle *K G O* is equal to the angle *I H O* (Theo. IV. Cor.). But, by hypothesis, the angle *K G O* is equal to the angle *C H O*, therefore the angle *I H O* is equal to *C H O*, so that *H I* and *H C* must coincide; that is, the line *C D* when produced meets *I K* in two points, *I*, *K*, and yet does not form one and the same straight line, which is impossible (4); therefore the lines *A B*, *C D* cannot meet, consequently they are parallel (Art. 14).

THEOREM XVI.

62. *If a straight line, intersecting two other straight lines, makes any exterior angle equal to the interior and opposite angle, or makes the interior angles on the same side together equal to two right angles, the two lines are parallel.*



Let the straight line *E F* intersect the two straight lines *A B*, *C D*, making the exterior angle *E G B* equal to the interior and opposite angle *G H D*; then the lines *A B*, *C D* are parallel.

For the angle *A G H* is equal to the angle *E G B* (Theo. III.); and *E G B* is equal to *G H D*, by hypothesis; therefore the angle *A G H* is equal to the angle *G H D*; and they are alternate angles; hence the lines *A B*, *C D* are parallel (Theo. XV.).

Again, let the interior angles on the same side, *B G H*, *G H D*, be together equal to two right angles; then the lines *A B*, *C D* are parallel.

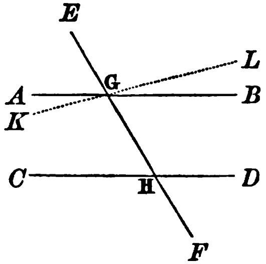
For the sum of the angles *B G H*, *G H D* is equal to two right angles, by hypothesis; and the sum of *A G H*, *B G H* is also equal to two right angles (Theo. I.); take away *B G H*, which is common to both, and there remains the angle *G H D*, equal to the angle *A G H;* and these are alternate angles; hence the lines *A B*, *C D* are parallel.

63. *Cor.* If two straight lines are perpendicular to another, they are parallel.

THEOREM XVII.

64. *If a straight line intersects two parallel lines, it makes the alternate angles equal; also any exterior angle equal to the interior and opposite angle; and the two interior angles upon the same side together equal to two right angles.*

Let the straight line *E F* intersect the parallel lines *A B*, *C D;* the alternate angles *A G H*, *G H D* are equal; the exterior angle *E G B* is equal to the interior and opposite angle *G H D;* and the two interior angles *B G H*, *G H D* upon the same side are together equal to two right angles.



For if the angle *A G H* is not equal to *G H D*, draw the straight line *K L* through the point *G*, making the angle *K G H* equal to *G H D;* then, since the alternate angles *G H D*, *K G H* are equal, *K L* is parallel to *C D* (Theo. XV.); but by hypothesis *A B* is also parallel to *C D*, so that through the same point, *G*, two straight lines are drawn parallel to *C D*, which is impossible. Hence the angles *A G H*, *G H D* are not unequal; that is, they are equal.

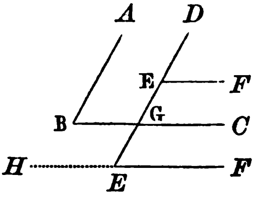
Now, the angle $E\,G\,B$ is equal to the angle $A\,G\,H$ (Theo. IV.), and $A\,G\,H$ has been shown to be equal to $G\,H\,D$; hence $E\,G\,B$ is also equal to $G\,H\,D$.

Again, add to each of these equals the angle $B\,G\,H$; then the sum of the angles $E\,G\,B$, $B\,G\,H$ is equal to the sum of the angles $B\,G\,H$, $G\,H\,D$. But $E\,G\,B$, $B\,G\,H$ are equal to two right angles (Theo. I.); hence $B\,G\,H$, $G\,H\,D$ are also equal to two right angles.

65. *Cor.* If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

THEOREM XVIII.

66. *If two angles have their sides parallel, each to each, and lying in the same direction, the two angles are equal.*



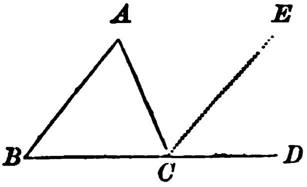
Let $A\,B\,C$, $D\,E\,F$ be two angles, which have the side $A\,B$ parallel to $D\,E$, and $B\,C$ parallel to $E\,F$; then these angles are equal.

For produce $D\,E$, if necessary, till it meets $B\,C$ in the point G. Then, since $E\,F$ is parallel to $G\,C$, the angle $D\,E\,F$ is equal to $D\,G\,C$ (Theo. XVII.); and since $D\,G$ is parallel to $A\,B$, the angle $D\,G\,C$ is equal to $A\,B\,C$; hence the angle $D\,E\,F$ is equal to $A\,B\,C$.

67. *Scholium.* This proposition is restricted to the case where the side $E\,F$ lies in the same direction with $B\,C$, since if $F\,E$ were produced toward H, the angles $D\,E\,H$, $A\,B\,C$ would only be equal when they are right angles.

THEOREM XIX.

68. *In every triangle the sum of the three angles is equal to two right angles.*



Let *A B C* be any triangle; then will the sum of the angles *A B C*, *B C A*, *C A B* be equal to two right angles.

For, draw *E C* parallel to the side *A B;* then, since *A C* meets the two parallels *A B*, *E C*, the alternate angles *B A C*, *A C E* are equal (Theo. XVII.).

Again, let the side *B C* be produced toward *D*, and since *B D* meets two parallels *A B*, *E C*, the exterior angle *E C D* is equal to the interior and opposite angle *A B C*. But the angle *A C E* is equal to *B A C;* therefore, the whole exterior angle *A C D* is equal to the two interior and opposite angles *C A B*, *A B C* (26, Ax. 2).

Add now to each of the equals *C A B* and *A B C* the angle *A C B*, and we shall have the sum of *A C B* and *A C D*, equal to the sum of *A B C*, *B C A*, and *C A B*. But the sum of *A C B* and *A C D* is equal to two right angles (Theo. I.); hence the sum of the three angles *A B C*, *B C A*, and *C A B* is equal to two right angles (26, Ax. 2).

69. *Cor.* 1. Two angles of a triangle being given, or merely their sum, the third will be found by subtracting that sum from two right angles.

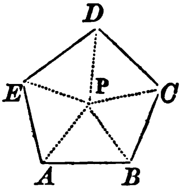
70. *Cor.* 2. If two angles in one triangle be respectively equal to two angles in another, their third angles will also be equal.

71. *Cor.* 3. A triangle cannot have more than one angle as great as a right angle.

72. *Cor.* 4. If any side of a triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles.

THEOREM XX.

73. *The sum of all the interior angles of any polygon is equal to twice as many right angles, less four, as the figure has sides.*



Let $A\,B\,C\,D\,E$ be any polygon; then the sum of all its interior angles, A, B, C, D, E, is equal to twice as many right angles as the figure has sides, less four right angles.

For, from any point P within the polygon, draw the straight lines $P\,A$, $P\,B$, $P\,C$, $P\,D$, $P\,E$, to the vertices of all the angles, and the polygon will be divided into as many triangles as it has sides. Now, the sum of the three angles in each of these triangles is equal to two right angles (Theo. XIX.); therefore the sum of the angles of all these triangles is equal to twice as many right angles as there are triangles, or sides, to the polygon. But the sum of all the angles about the point P is equal to four right angles (Theo. III. Cor. 2), which sum forms no part of the interior angles of the polygon; therefore, deducting the sum of the angles about the point, there remain the angles of the polygon equal to twice as many right angles as the figure has sides, less four right angles.

74. *Cor.* 1. The sum of the angles in a *quadrilateral* is equal to four right angles; hence, if all the angles of a quadrilateral are equal, each of them is a right angle; also, if three of the angles are right angles, the fourth is likewise a right angle.

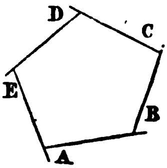
75. *Cor.* 2. The sum of the angles in a *pentagon* is equal to six right angles; in a *hexagon*, the sum is equal to eight right angles, &c.

76. *Cor.* 3. In every *equiangular* figure of more than four sides, each angle is greater than a right angle; thus, in a *regular pentagon*, each angle is equal to one and one fifth right angles; in a *regular hexagon*, to one and one third right angles, &c.

77. *Scholium.* This theorem is limited to *convex* polygons, or such as have their angles directed outwards; and, in general, by polygons, will be meant convex polygons.

THEOREM XXI.

78. *The sum of all the exterior angles of any polygon, formed by producing each side in the same direction, is equal to four right angles.*

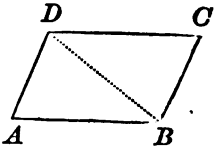


Let each side of the polygon *A B C D E* be produced in the same direction; then the sum of the exterior angles *A*, *B*, *C*, *D*, *E*, will be equal to four right angles.

For each interior angle, together with its adjacent exterior angle, is equal to two right angles (Theo. I.); hence the sum of all the angles, both interior and exterior, is equal to twice as many right angles as there are sides to the polygon. But the sum of the interior angles alone, is equal to the same sum, less four right angles (Theo. XX.); therefore the sum of the exterior angles is equal to four right angles.

THEOREM XXII.

79. *The opposite sides and angles of every parallelogram are equal to each other.*



Let *A B C D* be a parallelogram; then the opposite sides and angles are equal to each other.

Draw the diagonal *B D*, then, since the opposite sides *A B*, *D C* are parallel, and *B D* meets them, the alternate angles *A B D*, *B D C* are equal (Theo. XVII.); and since *A D*, *B C* are parallel, and *B D* meets them, the alternate angles *A D B*, *D B C* are likewise equal. Hence the two triangles *A D B*, *D B C* have two angles, *A B D*, *A D B*, in the one, equal to two angles, *B D C*, *D B C*, in the other, each to each; and since the side *B D* included between these equal angles is common to the two triangles, they are equal (Theo. V.), hence the side *A B* opposite the angle $A\,D\,B$ is equal to the side $D\,C$ opposite the angle $D\,B\,C$ (Theo. V. Cor.); and, in like manner, the side $A\,D$ is equal to the side $B\,C$; hence the opposite sides of a parallelogram are equal.

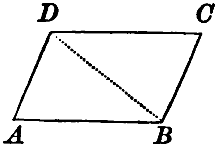
Again, since the triangles are equal, the angle A is equal to the angle C (Theo. V. Cor.); and since the two angles $D\,B\,C$, $A\,B\,D$ are respectively equal to the two angles $A\,D\,B$, $B\,D\,C$, the angle $A\,B\,C$ is equal to the angle $A\,D\,C$.

80. *Cor.* 1. The diagonal divides a parallelogram into two equal triangles.

81. *Cor.* 2. The two parallels $A\,D$, $B\,C$, included between two other parallels, $A\,B$, $C\,D$, are equal.

THEOREM XXIII.

82. *If the opposite sides of a quadrilateral are equal, each to each, the equal sides are parallel, and the figure is a parallelogram.*

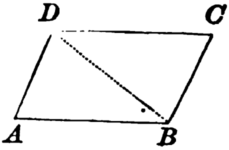


Let $A\,B\,C\,D$ be a quadrilateral having its opposite sides equal; then will the equal sides be parallel, and the figure be a parallelogram.

For, having drawn the diagonal $B\,D$, the triangles $A\,B\,D$, $B\,D\,C$ have all the sides of the one equal to the corresponding sides of the other; therefore they are equal, and the angle $A\,D\,B$ opposite the side $A\,B$ is equal to $D\,B\,C$ opposite $C\,D$ (Theo. XIII. Sch.); hence the side $A\,D$ is parallel to $B\,C$ (Theo. XV.). For a like reason, $A\,B$ is parallel to $C\,D$; therefore the quadrilateral $A\,B\,C\,D$ is a parallelogram.

THEOREM XXIV.

83. *If two opposite sides of a quadrilateral are equal and parallel, the other sides are also equal and parallel, and the figure is a parallelogram.*

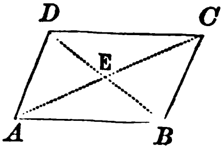


Let $ABCD$ be a quadrilateral, having the sides AB, CD equal and parallel; then will the other sides also be equal and parallel.

Draw the diagonal BD; then, since AB is parallel to CD, and BD meets them, the alternate angles ABD, BDC are equal (Theo. XVII.); moreover, in the two triangles ABD, DBC, the side BD is common; therefore, two sides and the included angle in the one are equal to two sides and the included angle in the other, each to each; hence these triangles are equal (Theo. IV.), and the side AD is equal to BC. Hence the angle ADB is equal to DBC, and consequently AD is parallel to BC (Theo. XV.); therefore the figure $ABCD$ is a parallelogram.

THEOREM XXV.

84. *The diagonals of every parallelogram bisect each other, or divide each other into equal parts.*

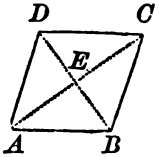


Let $ABCD$ be a parallelogram, and AC, DB its diagonals, intersecting at E; then will AE equal EC, and BE equal ED.

For, since AB, CD are parallel, and BD meets them, the alternate angles CDE, ABE are equal (Theo. XVII.); and since AC meets the same parallels, the alternate angles BAE, ECD are also equal; and the sides AB, CD are equal (Theo. XXII.). Hence the triangles ABE, CDE have two angles and the included side in the one equal to two angles and the included side in the other, each to each; hence the two triangles are equal (Theo. V.); therefore the side AE opposite the angle ABE is equal to CE opposite CDE; hence, also, the sides BE, DE opposite the other equal angles are equal.

85. *Scholium.* In the case of a rhombus, the sides AB, BC being equal, the triangles AEB, EBC have all the

sides of the one equal to the corresponding sides of the other, and are, therefore, equal; whence it follows that the angles *A E B*, *B E C* are equal. Therefore the diagonals of a rhombus bisect each other at right angles.



EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

86. 1. If the opposite angles formed by four lines meeting at a point are equal, these lines form but two straight lines.

2. If the equal sides of an isosceles triangle are produced, the two exterior angles formed with the base will be equal.

3. The line bisecting the vertical angle of an isosceles triangle bisects the base at right angles.

4. Every equilateral triangle is also equiangular.

5. Any side of a triangle is less than the sum of the other two.

6. The difference between any two sides of a triangle is less than the other side.

7. The side in a right-angled triangle opposite the right angle is the longest.

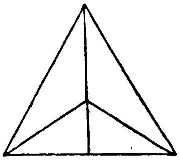
8. In every equilateral triangle each of the angles will be equal to two-thirds of one right angle.

9. Straight lines which are parallel to the same line are parallel to each other.

10. Lines joining the corresponding extremities of two equal and parallel straight lines, are themselves equal and parallel, and the figure formed is a parallelogram.

11. If, in the sides of a square, at equal distances from the four angles, points be taken, one in each side, the straight lines joining these points will form a square.

12. If from the middle point of a straight line a perpendicular be drawn, any point in the perpendicular will be equally distant from the extremities of the line.



BOOK II.

RATIO AND PROPORTION.

DEFINITIONS.

87. **Ratio** is the relation, in respect to quantity, which one magnitude bears to another of the same kind; and is the quotient arising from dividing the first by the second.

88. A ratio may be written in the form of a fraction, or with the sign : .

Thus the ratio of A to B may be expressed either by $\frac{A}{B}$, or by $A : B$.

89. The two magnitudes necessary to form a ratio are called the **Terms** of the ratio. The first term is called the **Antecedent**, and the last, the **Consequent**.

Ratios of magnitudes may be expressed by numbers either exactly, or approximately.

90. When the greater of two magnitudes contains the less a certain number of times without having a remainder, it is called a **Multiple** of the less; and the less is then called a **Submultiple**, or measure of the greater.

Thus, 6 is a multiple of 2; 2 and 3 are submultiples, or measures, of 6.

91. **Equimultiples**, or *like multiples*, are those which contain their respective submultiples the same number of times; and **Equisubmultiples**, or *like submultiples*, are those contained in their respective multiples the same number of times.

Thus 4 and 5 are like submultiples of 8 and 10; 8 and 10 are like multiples of 4 and 5.

Magnitudes of the same kind which have a common measure are said to be *commensurable;* and those which have no common measure are said to be *incommensurable.*

92. A **Direct Ratio** is the quotient of the antecedent by the consequent; an **Inverse Ratio**, or **Reciprocal Ratio**, is the quotient of the consequent by the antecedent, or the reciprocal of the direct ratio.

Thus the direct ratio of a line 6 feet long to a line 2 feet long is $\frac{6}{2}$ or 3; and the inverse ratio of a line 6 feet long to a line 2 feet long is $\frac{2}{6}$ or $\frac{1}{3}$, which is the same as the reciprocal of 3, the direct ratio of 6 to 2.

The word *ratio* when used alone means the direct ratio.

93. A **Compound Ratio** is the product of two or more ratios.

Thus the ratio compounded of $A : B$ and $C : D$ is $\frac{A}{B} \times \frac{C}{D}$, or $\frac{A \times C}{B \times D}$.

94. A **Proportion** is an equality of ratios.

Four magnitudes are in proportion, when the ratio of the *first* to the *second* is the same as that of the *third* to the *fourth.*

Thus, the ratios of $A : B$ and $X : Y$, being equal to each other, when written $A : B = X : Y$, or $\frac{A}{B} = \frac{X}{Y}$, form a proportion.

95. **Proportion** is written not only with the sign $=$, but, more often, with the sign $::$ between the ratios.

Thus, $A : B :: X : Y$, expresses a proportion, and is read, The ratio of A to B is equal to the ratio of X to Y; or, A is to B as X is to Y.

96. The *first* and *third* terms of a proportion are called the **Antecedents**; the *second* and *fourth*, the **Consequents.** The *first* and *fourth* are also called the **Extremes, and the** *second* and *third* the **Means.**

Thus, in the proportion $A : B :: C : D$, A and C are the antecedents; B and C are the consequents; A and D are the extremes; and B and C are the means.

The antecedents are called *homologous* or *like* terms, and so also are the consequents.

97. All the terms of a proportion are called **Proportionals**; and the last term is called a *fourth proportional* to the other three taken in their order.

Thus, in the proportion $A : B :: C : D$, D is the fourth proportional to A, B, and C.

98. When both the means are the same magnitude, either of them is called a *mean proportional* between the extremes; and if, in a series of proportional magnitudes, each consequent is the same as the next antecedent, those magnitudes are said to be in *continued proportion*.

Thus, if we have $A : B :: B : C :: C : D :: D : E$, B is a mean proportional between A and C, C between B and D, D between C and E; and the magnitudes A, B, C, D, E are said to be in continued proportion.

99. When a continued proportion consists of but three terms, the middle term is said to be a *mean proportional* between the other two; and the last term is said to be the *third proportional* to the first and second.

Thus, when A, B, and C are in proportion, $A : B :: B : C$; in which case B is called a mean proportional between A and C; and C is called the third proportional to A and B.

100. Magnitudes are in proportion by **Inversion**, or *inversely*, when each antecedent takes the place of its consequent, and each consequent the place of its antecedent.

Thus, let $A : B :: C : D$; then, by inversion,

$$B : A :: D : C.$$

101. Magnitudes are in proportion by **Alternation**, or *alternately*, when antecedent is compared with antecedent, and consequent with consequent.

Thus, let $A : B :: D : C$; then, by alternation,

$$A : D :: B : C.$$

102. Magnitudes are in proportion by **Composition**, when the sum of the first antecedent and consequent is to the first antecedent, or consequent, as the sum of the second antecedent and consequent is to the second antecedent, or consequent.

Thus, let $A : B :: C : D$; then, by composition,

$$A+B : A :: C+D : C, \text{ or } A+B : B :: C+D : D.$$

103. Magnitudes are in proportion by **Division**, when the difference of the first antecedent and consequent is to the first antecedent, or consequent, as the difference of the second antecedent and consequent is to the second antecedent, or consequent.

Thus, let $A : B :: C : D$; then, by division,

$$A-B : A :: C-D : C, \text{ or } A-B : B :: C-D : D.$$

THEOREM I.

104. *If four magnitudes are in proportion, the product of the two extremes is equal to the product of the two means.*

Let $A : B :: C : D$; then will $A \times D = B \times C$.

For, since the magnitudes are in proportion,

$$\frac{A}{B} = \frac{C}{D};$$

and reducing the fractions of this equation to a common denominator, we have

$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D},$$

or, the common denominator being omitted,

$$A \times D = B \times C.$$

THEOREM II.

105. *If the product of two magnitudes is equal to the product of two others, these four magnitudes form a proportion.*

Let $A \times D = B \times C$; then will $A : B :: C : D$.

For, dividing each member of the given equation by $B \times D$, we have

$$\frac{A \times D}{B \times D} = \frac{B \times C}{B \times D},$$

which, reduced to the lowest terms, gives

$$\frac{A}{B} = \frac{C}{D}.$$

Whence $A : B :: C : D$.

THEOREM III.

106. *If the product of any two quantities is equal to the square of a third, the third is a mean proportional between the other two.*

Let $A \times C = B^2$; then B is a mean proportional between A and C.

For, dividing each member of the given equation by $B \times C$, we have

$$\frac{A}{B} = \frac{B}{C},$$

whence $A : B :: B : C$.

THEOREM IV.

107. *If four magnitudes are in proportion, they will be in proportion when taken inversely.*

Let $A : B :: C : D$; then will $B : A :: D : C$.

For, from the given proportion, by Theo. I., we have

$$A \times D = B \times C, \text{ or } B \times C = A \times D.$$

Hence, by Theo. II.,

$$B : A :: D : C.$$

THEOREM V.

108. *If four magnitudes are in proportion, they will be in proportion when taken alternately.*

Let $A : B :: C : D$; then will $A : C :: B : D$.

For, since the magnitudes are in proportion,

$$\frac{A}{B}=\frac{C}{D};$$

and multiplying each member of this equation by $\frac{B}{C}$, we have

$$\frac{A\times B}{B\times C}=\frac{C\times B}{D\times C},$$

which, reduced to the lowest terms, gives

$$\frac{A}{C}=\frac{B}{D},$$

whence $$A : C :: B : D.$$

THEOREM VI.

109. *If four magnitudes are in proportion, they will be in proportion by composition.*

Let $A : B :: C : D$; then will $A+B : A :: C+D : C$.

For, from the given proportion, by Theo. I., we have

$$B\times C=A\times D.$$

Adding $A\times C$ to each side of this equation, we have

$$A\times C+B\times C=A\times C+A\times D,$$

and resolving each member into its factors,

$$(A+B)\times C=(C+D)\times A.$$

Hence, by Theo. II.,

$$A+B : A :: C+D : C.$$

THEOREM VII.

110. *If four magnitudes are in proportion, they will be in proportion by division.*

Let $A : B :: C : D$; then will $A-B : A :: C-D : C$.

For, from the given proportion, by Theo. I., we have

$$B\times C=A\times D.$$

Subtracting each side of this equation from $A \times C$, we have

$$A \times C - B \times C = A \times C - A \times D,$$

and resolving each member into its factors,

$$(A - B) \times C = (C - D) \times A.$$

Hence, by Theo. II.,

$$A - B : A :: C - D : C.$$

THEOREM VIII.

111. *Equimultiples of two magnitudes have the same ratio as the magnitudes themselves.*

Let A and B be two magnitudes, and $m \times A$ and $m \times B$ their equimultiples, then will $m \times A : m \times B :: A : B$.

For $$A \times B = B \times A.$$

Multiplying each side of this equation by any number, m, we have

$$m \times A \times B = m \times B \times A;$$

therefore

$$(m \times A) \times B = (m \times B) \times A.$$

Hence, by Theo. II.,

$$m \times A : m \times B :: A : B.$$

THEOREM IX.

112. *Magnitudes which are proportional to the same proportionals, will be proportional to each other.*

Let $A : B :: E : F$, and $C : D :: E : F$; then will

$$A : B :: C : D.$$

For, by the given proportions, we have

$$\frac{A}{B} = \frac{E}{F} \text{ and } \frac{C}{D} = \frac{E}{F}.$$

Therefore, it is evident (26, Ax. 1),

$$\frac{A}{B} = \frac{C}{D}.$$

Hence $$A : B :: C : D.$$

113. *Cor.* 1. If two proportions have an antecedent and its consequent the same in both, the remaining terms will be in proportion.

114. *Cor.* 2. Therefore, by alternation (Theo. V.), if two proportions have the two antecedents or the two consequents the same in both, the remaining terms will be in proportion.

THEOREM X.

115. *If any number of magnitudes are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $A : B :: C : D :: E : F$; then will

$$A : B :: A + C + E : B + D + F.$$

For, from the given proportion, we have

$$A \times D = B \times C, \text{ and } A \times F = B \times E.$$

By adding $A \times B$ to the sum of the corresponding sides of these equations, we have

$$A \times B + A \times D + A \times F = A \times B + B \times C + B \times E.$$

Therefore,

$$A \times (B + D + F) = B \times (A + C + E).$$

Hence, by Theo. II.,

$$A : B :: A + C + E ; B + D + F.$$

THEOREM XI.

116. *If there be two sets of proportional magnitudes, the products of the corresponding terms will be proportionals.*

Let $A : B :: C : D$, and $E : F :: G : H$; then will

$$A \times E : B \times F :: C \times G : D \times H.$$

For, from the first of the given proportions, by Theo. I., we have

$$A \times D = B \times C;$$

and from the second of the given proportions, by Theo. I., we have

$$E \times H = F \times G.$$

Multiplying together the corresponding members of these equations, we have

$$A \times D \times E \times H = B \times C \times F \times G.$$

Hence, by Theo. II.,

$$A \times E : B \times F :: C \times G : D \times H.$$

THEOREM XII.

117. *If four magnitudes are proportionals, their like powers and roots will also be proportional.*

Let $A : B :: C : D$; then will

$$A^n : B^n :: C^n : D^n, \text{ and } A^{\frac{1}{n}} : B^{\frac{1}{n}} :: C^{\frac{1}{n}} : D^{\frac{1}{n}}.$$

For, from the given proportion, we have

$$\frac{A}{B} = \frac{C}{D}.$$

Raising both members of this equation to the nth power, we have

$$\frac{A^n}{B^n} = \frac{C^n}{D^n},$$

and extracting the nth root of each member, we have

$$\frac{A^{\frac{1}{n}}}{B^{\frac{1}{n}}} = \frac{C^{\frac{1}{n}}}{D^{\frac{1}{n}}}.$$

Hence, by Theo. II., the last two equations give

$$A^n : B^n :: C^n : D^n,$$

and

$$A^{\frac{1}{n}} : B^{\frac{1}{n}} :: C^{\frac{1}{n}} : D^{\frac{1}{n}}.$$

EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

118. 1. If three magnitudes are in proportion, the product of the two extremes is equal to the square of the mean.

2. If four magnitudes are proportionals, the first and second may be multiplied or divided by the same magnitude, and also the third and fourth by the same magnitude, and the resulting magnitudes will be proportionals.

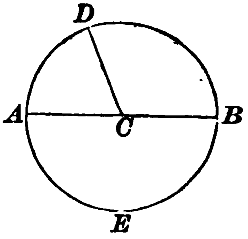
3. If four magnitudes are proportionals, the first and third may be multiplied or divided by the same magnitude, and also the second and fourth by the same magnitude, and the resulting magnitudes will be proportionals.

4. If there be two sets of proportional magnitudes, the quotients of the corresponding terms will be proportionals.

BOOK III.

THE CIRCLE, AND THE MEASURE OF ANGLES.

DEFINITIONS.



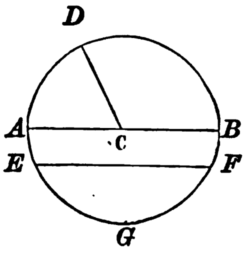
119. A **Circle** is a plane figure bounded by a curved line, all the points of which are equally distant from a point within called the *center;* as the figure *A D B E*.

120. The **Circumference** or *periphery* of a circle is its entire bounding line; or it is a curved line, all points of which are equally distant from a point within called the center.

121. A **Radius** of a circle is any straight line drawn from the center to the circumference; as the line *C A*, *C D*, or *C B*.

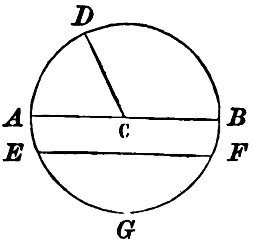
122. A **Diameter** of a circle is any straight line drawn through the center and terminating in both directions in the circumference; as the line *A B*.

123. All the radii of a circle are equal; all the diameters are also equal, and each is double the radius.



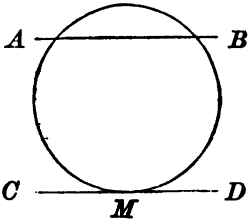
124. An **Arc** of a circle is any part of the circumference; as the part *A D*, *A E*, or *E G F*.

125. The **Chord** of an arc is the straight line joining its extremities; thus *E F* is the chord of the arc *E G F*.



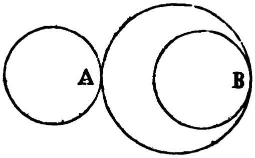
126. The **Segment** of a circle is the part of a circle included between an arc and its chord; as the surface included between the arc *E G F* and the chord *E F*.

127. The **Sector** of a circle is the part of a circle included between an arc, and the two radii drawn to the extremities of the arc; as the surface included between the arc *A D*, and the two radii *C A*, *C D*.

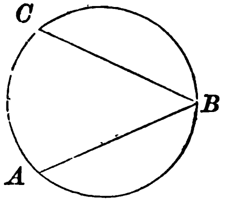


128. A **Secant** to a circle is a straight line which cuts the circumference in two points, and lies partly within and partly without the circle; as the line *A B*.

129. A **Tangent** to a circle is a straight line which, how far so ever produced, meets the circumference in but one point; as the line *C D*. The point of meeting is called the *point of contact;* as the point *M*.

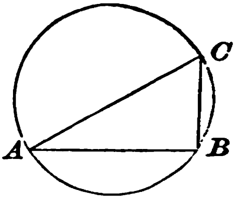


130. Two circumferences *touch* each other, when they have a point of contact without cutting one another; thus two circumferences touch each other at the point *A*, and two at the point *B*.



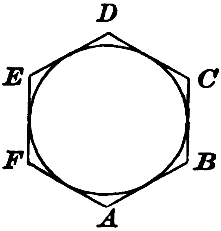
131. A straight line is *inscribed* in a circle when its extremities are in the circumference; as the line *A B*, or *B C*.

132. An **Inscribed Angle** is one which has its vertex in the circumference, and is formed by two chords; as the angle *A B C*.



133. An Inscribed Polygon is one which has the vertices of all its angles in the circumference of the circle; as the triangle $A\,B\,C$.

134. The circle is then said to be *circumscribed* about the polygon.

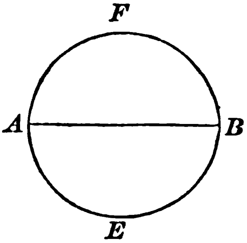


135. A Polygon is circumscribed about a circle when all its sides are tangents to the circumference; as the polygon $A\,B\,C\,D\,E\,F$.

136. The circle is then said to be *inscribed* in the polygon.

THEOREM I.

137. *Every diameter divides the circle and its circumference each into two equal parts.*



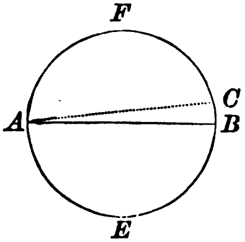
Let $A\,E\,B\,F$ be a circle, and $A\,B$ a diameter; then the two parts $A\,E\,B$, $A\,F\,B$ are equal.

For, if the figure $A\,E\,B$ be applied to $A\,F\,B$, their common base $A\,B$ retaining its position, the curve line $A\,E\,B$ must fall exactly on the curve line $A\,F\,B$; otherwise there would be points in the one or the other unequally distant from the center, which is contrary to the definition of the circle.

138. Hence a diameter divides the circle and its circumference into two equal parts.

139. *Cor.* 1. *Conversely,* a straight line dividing the circle into two equal parts is a diameter.

For, let the line $A\,B$ divide the circle $A\,E\,B\,C\,F$ into two equal parts; then, if the center is not in $A\,B$, let $A\,C$ be



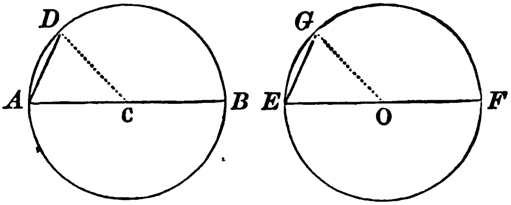
drawn through it, which is therefore a diameter, and consequently divides the circle into two equal parts; hence the surface *A F C* is equal to the surface *A F C B*, a part to the whole, which is impossible.

140. *Cor.* 2. The arc of a circle, whose chord is a diameter, is a semi-circumference, and the included segment is a semicircle.

Theorem II.

141. *In the same circle, or in equal circles, equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.*

Let *A D B* and *E G F* be two equal circles, and let the arc *A D* be equal to *E G;* then will the chord *A D* be equal to the chord *E G*.



For, since the diameters *A B*, *E F* are equal, the semicircle *A D B* may be applied to the semicircle *E G F;* and the curve line *A D B* will coincide with the curve line *E G F* (Theo. I.). But, by hypothesis, the arc *A D* is equal to the arc *E G;* hence the point *D* will fall on *G;* hence the chord *A D* is equal to the chord *E G* (26, Ax. 10).

Conversely, if the chord *A D* is equal to the chord *E G*, the arcs *A D*, *E G* will be equal.

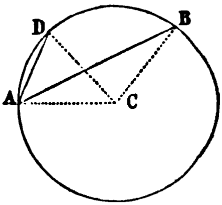
For, if the radii *C D*, *O G* are drawn, the triangles *A C D*, *E O G*, having the three sides of the one equal to the three sides of the other, each to each, are themselves equal (Theo. XIII. Bk. I.); therefore the angle *A C D* is equal to the angle *E O G* (Theo. XIII. Sch., Bk. I.).

If now the semicircle ADB be applied to its equal EGF, with the radius AC on its equal EO, since the angles ACD, EOG are equal, the radius CD will fall on OG, and the point D on G. Therefore the arcs AD and EG coincide with each other; hence they must be equal (26, Ax. 12).

THEOREM III.

142. *In the same circle, or in equal circles, a greater arc is subtended by a greater chord; and, conversely, the greater chord subtends the greater arc.*

In the circle of which C is the center, let the arc AB be greater than the arc AD; then will the chord AB be greater than the chord AD.



Draw the radii CA, CD, and CB. The two sides AC, CB in the triangle ACB are equal to the two AC, CD in the triangle ACD, and the angle ACB is greater than the angle ACD; therefore the third side AB is greater than the third side AD (Theo. XI. Bk. I.); hence the chord which subtends the greater arc is the greater.

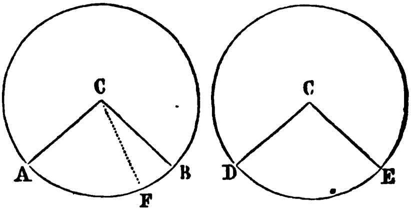
Conversely, if the chord AB be greater than the chord AD, the arc AB will be greater than the arc AD.

For the triangles ACB, ACD have two sides, AC, CB, in the one, equal to two sides, AC, CD, in the other, while the side AB is greater than the side AD; therefore the angle ACB is greater than the angle ACD (Theo. XII. Bk. I.); hence the arc AB is greater than the arc AD.

143. *Scholium.* The arcs here treated of are each less than the semi-circumference. If they were greater, the contrary would be true; in which case, as the arcs increased, the chords would diminish, and conversely.

THEOREM IV.

144. *In the same circle, or in equal circles, radii which make equal angles at the center intercept equal arcs on the circumference; and, conversely, if the intercepted arcs are equal, the angles made by the radii are also equal.*



Let $A\,C\,B$ and $D\,C\,E$ be equal angles made by radii at the center of equal circles; then will the intercepted arcs $A\,B$ and $D\,E$ be also equal.

First. Since the angles $A\,C\,B$, $D\,C\,E$ are equal, the one may be applied to the other; and since their sides, being radii of equal circles, are equal, the point A will coincide with D, and the point B with E. Therefore the arc $A\,B$ must also coincide with the arc $D\,E$, or there would be points in the one or the other unequally distant from the center, which is impossible; hence the arc $A\,B$ is equal to the arc $D\,E$.

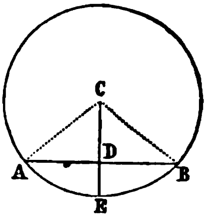
Second. If the arcs $A\,B$ and $D\,E$ are equal, the angles $A\,C\,B$ and $D\,C\,E$ will be equal.

For, if these angles are not equal, let $A\,C\,B$ be the greater, and let $A\,C\,F$ be taken equal to $D\,C\,E$. From what has been shown, we shall have the arc $A\,F$ equal to the arc $D\,E$. But, by hypothesis, $A\,B$ is equal to $D\,E$; hence $A\,F$ must be equal to $A\,B$, the part to the whole, which is impossible; hence the angle $A\,C\,B$ is equal to the angle $D\,C\,E$.

THEOREM V.

145. *The radius which is perpendicular to a chord bisects the chord, and also the arc subtended by the chord.*

Let the radius $C\,E$ be perpendicular to the chord $A\,B$; then will $C\,E$ bisect the chord at D, and the arc $A\,B$ at E.



Draw the radii CA and CB. Then CA and CB, with respect to the perpendicular CE, are equal oblique lines drawn to the chord AB; therefore their extremities are at equal distances from the perpendicular (Theo. X. Bk. I.); hence AD and DB are equal.

Again, since the triangle ACB has the sides AC and CB equal, it is isosceles; and the line CE bisects the base AB at right angles; therefore CE bisects also the angle ACB (Theo. VI. Book I.). Since the angles ACD, DCB are equal, the arcs AE, EB are equal (Theo. IV.); hence the radius CE, which is perpendicular to the chord AB, bisects the arc AB subtended by the chord.

146. *Cor.* 1. Any straight line which joins the center of the circle and the middle of the chord, or the middle of the arc, must be perpendicular to the chord.

For, the perpendicular from the center C passes through the middle, D, of the chord, and the middle, E, of the arc subtended by the chord. Now, any two of these three points in the straight line CE, are sufficient to determine its position.

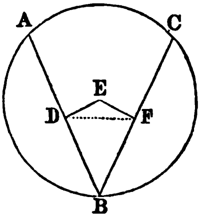
147. *Cor.* 2. A perpendicular at the middle of a chord passes through the center of the circle, and through the middle of the arc subtended by the chord, bisecting at the center the angle which the arc subtends.

THEOREM VI.

148. *Through three given points, not in the same straight line, one circumference can be made to pass, and but one.*

Let A, B, and C be any three points not in the same straight line; one circumference can be made to pass through them, and but one.

Join AB and BC; and bisect these straight lines by the



perpendiculars *D E* and *F E*. Join *D F;* then, the angles *B D E*, *B F E*, being each a right angle, are together equal to two right angles; therefore the angles *E D F*, *E F D* are together less than two right angles; and *D E*, *F E*, produced, must meet in some point *E*.

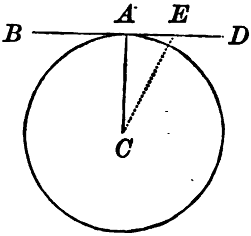
Now, since the point *E* lies in the perpendicular *D E*, it is equally distant from the two points *A* and *B* (Theo. X. Bk. I.); and since the same point *E* lies in the perpendicular *F E*, it is also equally distant from the two points *B* and *C;* therefore the three distances, *E A*, *E B*, *E C*, are equal; hence a circumference can be described from the center *E* passing through the three points *A*, *B*, *C*.

Again, the center, lying in the perpendicular *D E* bisecting the chord *A B*, and at the same time in the perpendicular *F E* bisecting the chord *B C* (Theo. V. Cor. 2), must be at the point of their meeting, *E*. Therefore, since there can be but one center, but one circumference can be made to pass through three given points.

149. *Cor.* Two circumferences can intersect in only two points; for, if they have three points in common, they must have the same center, and must coincide.

THEOREM VII.

150. *A straight line perpendicular to a radius at its termination in the circumference, is a tangent to the circle.*



Let the straight line *B D* be perpendicular to the radius *C A* at its termination *A;* then will it be a tangent to the circle.

Draw from the center *C* to *B D* any other straight line, as *C E*. Then, since *C A* is perpendicular to *B D*, it is shorter than the oblique line *C E* (Theo. X. Bk. I.); hence the point *E* is without the circle. The same may be shown of any other point in the line *B D*, except the point *A;* therefore *B D* meets the circumference at *A*, and, being produced, does not cut it; hence *B D* is a tangent (129).

151. *Cor.* 1. Conversely, if a line is a tangent to a circumference, the radius drawn to the point of contact with it is perpendicular to the tangent.

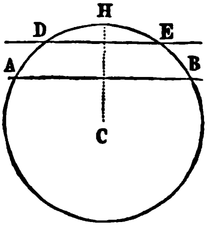
For every point in *B D*, except *A*, being without the circumference, any line *C E* drawn from the center, *C*, to *B D*, at any point other than *A*, must terminate at *E*, without the circumference; therefore the radius *C A* is the shortest line that can be drawn from the center to *B D;* hence *C A* is perpendicular to the tangent *B D* (Theo. X. Cor. 1, Bk. I.).

152. *Cor.* 2. Only one tangent can be drawn through the same point in a circumference; for two lines cannot both be perpendicular to a radius at the same point.

THEOREM VIII.

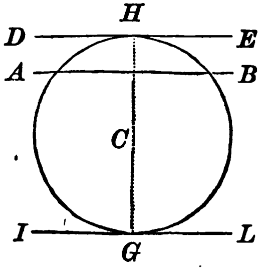
153. *Two parallel straight lines intercept equal arcs of the circumference.*

First. When the two parallels are secants, as *A B*, *D E*.



Draw the radius *C H* perpendicular to *A B;* and it will also be perpendicular to *D E* (Theo. XVII. Cor., Bk. I.); therefore the point *H* will be at the same time the middle of the arc *A H B* and of the arc *D H E* (Theo. V.); therefore, the arc *A H* is equal to the arc *H B*, and the arc *D H* is equal to the arc *H E;* hence *A H* diminished by *D H* is equal to *H B* diminished by *H E;* that is, the intercepted arcs *A D*, *B E* are equal.

Second. When of the two parallels, one, as *A B*, is a secant, and the other, as *D E*, is a tangent.



Draw the radius *C H* to the point of contact *H*. This radius will be perpendicular to the tangent *D E* (Theo. VII.), and also to its parallel *A B* (Theo. XVI. Cor., Bk. I.). But, since *C H* is perpendicular to the chord *A B*, the point *H* is the middle of the arc *A H B;* hence the arcs *A H*, *H B*, included between the parallels *A B*, *D E*, are equal.

Third. When the two parallels are tangents, as *D E*, *I L*.

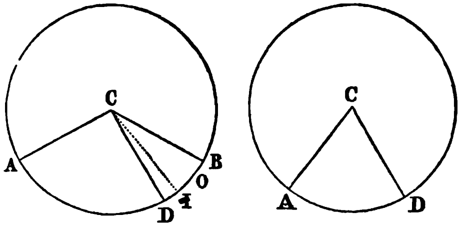
Draw the secant *A B* parallel to either of the tangents; then, from what has been just shown, the arc *A H* is equal to the arc *H B*, and also the arc *A G* is equal to the arc *G B;* hence the whole arc *H A G* is equal to the whole arc *H B G*.

It is further evident, since the two arcs *H A G*, *H B G* are equal, and together make up the whole circumference, that each of them is a semi-circumference.

Theorem IX.

154. *In the same circle, or in equal circles, any two angles at the center are to each other as the arcs intercepted between their sides.*

Let *A C B* be the greater, and *A C D* the less angle; then will the angle *A C B* be to the angle *A C D* as the arc *A B* is to the arc *A D*.



Conceive the less angle to be placed on the greater; then, if the proposition be not true, the angle *A C B* will be to the angle *A C D* as the arc *A B* is to an arc greater or less than

A D. Suppose this arc to be greater, and let it be represented by *A O;* we shall have the angle *A C B* : angle *A C D* : : arc *A B* : arc *A O*. Conceive, now, the arc *A B* to be divided into equal parts, each of which is less than *D O;* there will be at least one point of division between *D* and *O;* let *I* be that point; and join *C I*. The arcs *A B*, *A I* will be to each other as two whole numbers, and, by the preceding proposition, we shall have the angle *A C B* : angle *A C I* : : arc *A B* : arc *A I*. Comparing these two proportions with each other, and observing that the antecedents are the same, we infer that the consequents are proportional (Theo. IX. Cor. 2, Bk. II.); hence the angle *A C D* : angle *A C I* : : arc *A O* : arc *A I*. But the arc *A O* is greater than the arc *A I;* therefore, if this proportion is true, the angle *A C D* must be greater than the angle *A C I*. But it is less; hence the angle *A C B* cannot be to the angle *A C D* as the arc *A B* is to an arc greater than *A D*.

By a process of reasoning entirely similar, it may be shown that the fourth term of the proportion cannot be less than *A D;* therefore it must be *A D;* hence we have,

Angle *A C B* : angle *A C D* : : arc *A B* : arc *A D*.

155. *Cor.* 1. An angle having its vertex at the center of a circle is measured by the arc included between its sides.

For the angle at the center of a circle, and the arc intercepted by its sides, have such a connection, that, if the one be increased or diminished in any ratio, the other will be increased or diminished in the same ratio, we are authorized to take the one of these magnitudes as the measure of the other.

156. *Cor.* 2. Four right angles are measured by an entire circumference (Theo. III. Cor. 2, Bk. I.).

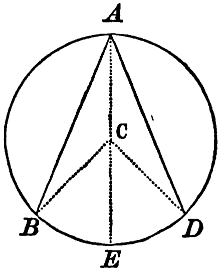
157. *Cor.* 3. Hence, if a right angle is measured by a quadrant, or 90 degrees; two right angles by a semi-circumference, or 180 degrees; and four right angles by the entire circumference, or 360 degrees.

158. *Scholium* 1. In the comparison of angles with each other, the arcs which serve to measure them must be described with equal radii.

159. *Scholium* 2. Sectors taken in the same circle, or in equal circles, are to each other as their arcs; for sectors are equal when their angles are so, and therefore are in all respects proportional to their angles.

THEOREM X.

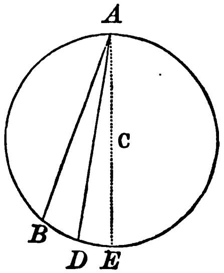
160. *An inscribed angle is measured by half the arc included between its sides.*



Let *B A D* be an inscribed angle, whose sides include the arc *B D;* then the angle *B A D* is measured by half of the arc *B D.*

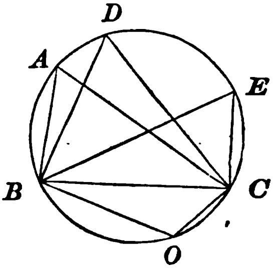
First. Suppose the center of the circle *C* to lie within the angle *B A D.* Draw the diameter *A E*, and the radii *C B*, *C D.*

The angle *B C E*, being exterior to the triangle *A B C*, is equal to the sum of the two interior angles *C A B*, *A B C* (Theo. XIX. Cor. 4, Bk. I). But the triangle *B A C* being isosceles, the angle *C A B* is equal to *A B C;* hence, the angle *B C E* is double *B A C.* Since *B C E* lies at the center, it is measured by the arc *B E* (Theo. IX. Cor. 1); hence *B A C* will be measured by half of *B E.* For a like reason, the angle *C A D* will be measured by the half of *E D;* hence *B A C* and *C A D* together, or *B A D*, will be measured by the half of *B E* and *E D*, or half *B D.*

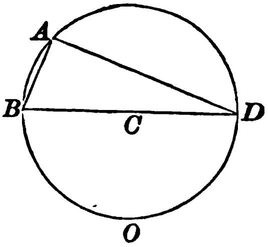


Second. Suppose that the center *C* lies without the angle *B A D.* Then, drawing the diameter *A E*, the angle *B A E* will be measured by the half of *B E;* and the angle *D A E* is measured by the half of *D E;* hence, their difference, $B\,A\,D$, will be measured by the half of $B\,E$ *minus* the half of $E\,D$, or by the half of $B\,D$.

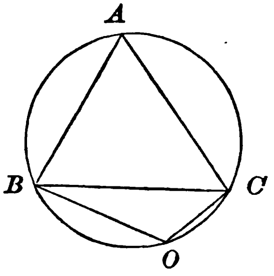
Hence every inscribed angle is measured by the half of the arc included between its sides.



161. *Cor.* 1. All the angles, $B\,A\,C$, $B\,D\,C$, inscribed in the same segment, are equal; because they are all measured by the half of the same arc, $B\,O\,C$.

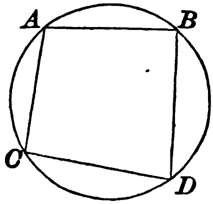


162. *Cor.* 2. Every angle, $B\,A\,D$, inscribed in a semicircle, is a right angle; because it is measured by half the semi-circumference, $B\,O\,D$; that is, by the fourth part of the whole circumference.



163. *Cor.* 3. Every angle, $B\,A\,C$, inscribed in a segment greater than a semicircle, is an acute angle; for it is measured by the half of the arc $B\,O\,C$, less than a semi-circumference.

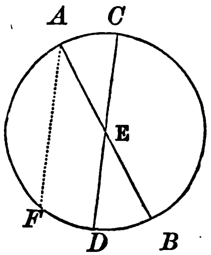
And every angle, $B\,O\,C$, inscribed in a segment less than a semicircle, is an obtuse angle; for it is measured by half of the arc $B\,A\,C$, greater than a semi-circumference.



164. *Cor.* 4. The opposite angles, A and D, of an inscribed quadrilateral, $A\,B\,D\,C$, are together equal to two right angles; for the angle $B\,A\,C$ is measured by half the arc $B\,D\,C$, and the angle $B\,D\,C$ is measured by half the arc $B\,A\,C$; hence the two angles $B\,A\,C$, $B\,D\,C$, taken together, are measured by half the circumference; hence their sum is equal to two right angles.

THEOREM XI.

165. *The angle formed by the intersection of two chords is measured by half the sum of the two intercepted arcs.*

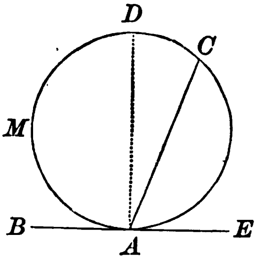


Let the two chords *A B*, *C D* intersect each other at the point *E;* then will the angle *D E B*, or its equal, *A E C*, be measured by half the sum of the two arcs *D B* and *A C*.

Draw *A F* parallel to *D C;* then will the arc *F D* be equal to the arc *A C* (Theo. VIII.), and the angle *F A B* equal to the angle *D E B* (Theo. XVII. Bk. I.). But the angle *F A B* is measured by half the arc *F D B* (Theo. X.); that is, by half the arc *D B*, plus half the arc *F D*. Hence, since *F D* is equal to *A C*, the angle *D E B*, or its equal angle *A E C*, is measured by half the sum of the intercepted arcs *D B* and *A C*.

THEOREM XII.

166. *The angle formed by a tangent and a chord is measured by half the intercepted arc.*



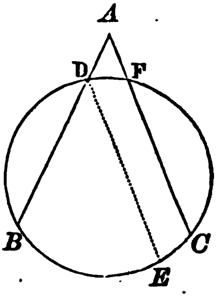
Let the tangent *B E* form, with the chord *A C*, the angle *B A C;* then *B A C* is measured by half the arc *A M C*.

From *A*, the point of contact, draw the diameter *A D*. The angle *B A D* is a right angle (Theo. VII.), and is measured by half of the semi-circumference *A M D* (Theo. X.); and the angle *D A C* is measured by half the arc *D C;* hence the sum of the angles *B A D*, *D A C*, or *B A C*, is measured by the half of *A M D*, plus the half of *D C;* or by half the whole arc *A M D C.*

In like manner, it may be shown that the angle *C A E* is measured by half the intercepted arc *A C.*

THEOREM XIII.

167. *The angle formed by two secants is measured by half the difference of the two intercepted arcs.*

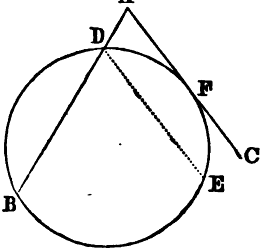


Let *A B*, *A C* be two secants forming the angle *B A C;* then will that angle be measured by half the difference of the two arcs *B E C* and *D F.*

Draw *D E* parallel to *A C;* then will the arc *E C* be equal to the arc *D F* (Theo. VIII.); and the angle *B D E* be equal to the angle *B A C* (Theo. XVII. Bk. I.). But the angle *B D E* is measured by half the arc *B E* (Theo. X.); hence the equal angle *B A C* is also measured by half the arc *B E;* that is, by half the difference of the arcs *B E C* and *E C*, or, since *E C* is equal to *D F*, by half the difference of the intercepted arcs *B E C* and *D F.*

THEOREM XIV.

168. *The angle formed by a secant and a tangent is measured by half the difference of the two intercepted arcs.*



Let the secant *A B* form, with the tangent *A C*, the angle *B A C;* then *B A C* is measured by half the difference of the two arcs *B E F* and *F D.*

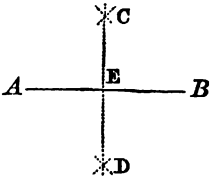
Draw *D E* parallel to *A C;* then will the arc *E F* be equal to the arc *D F* (Theo. VIII.), and the angle

B D E be equal to the angle *B A C*. But the angle *B D E* is measured by half of the arc *B E* (Theo. X.); hence the equal angle *B A C* is also measured by half the arc *B E;* that is, by half the difference of the arcs *B E F* and *E F*, or, since *E F* is equal to *D F*, by half the difference of the intercepted arcs *B E F* and *D F*.

PROBLEMS IN CONSTRUCTION.

169. **The Problems in Construction,** which follow in this book, require for their solution the application of the principles of the preceding books.

PROBLEM I. *To bisect a given straight line, or to divide it into two equal parts.*



Let *A B* be a straight line, which it is required to bisect.

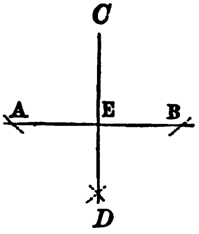
From the point *A* as a center, with a radius greater than the half of *A B*, describe an arc of a circle; and from the point *B* as a center, with the same radius, describe another arc, cutting the former in the points *C* and *D*. Through *C* and *D* draw the straight line *C D;* it will bisect *A B* in the point *E*.

For the two points *C* and *D*, being each equally distant from the extremities *A* and *B*, must both lie in the perpendicular raised from the middle point of *A B* (Theo. X. Cor. 2, Bk. I.). Therefore the line *C D* must divide the line *A B* into two equal parts at the point *E*.

PROBLEM II. *From a given point, without a straight line, to draw a perpendicular to that line.*

Let *A B* be the straight line, and let *C* be a given point without the line.

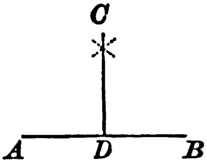
From the point *C* as a center, and with a radius sufficiently great, describe an arc cutting the line *A B* in two points, *A* and *B;* then, from the points *A* and *B* as centers, with a radius greater than half of $A\,B$, describe two arcs cutting each other in D, and draw the straight line $C\,D$; it will be the perpendicular required.



For, the two points C and D are each equally distant from the points A and B; hence the line $C\,D$ is a perpendicular passing through the middle of $A\,B$ (Theo. X. Cor. 2, Bk. I.).

Problem III. *At a given point in a straight line to erect a perpendicular to that line.*

Let $A\,B$ be the straight line, and let D be a given point in it.



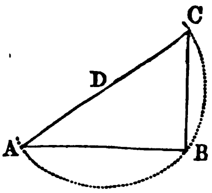
In the straight line $A\,B$, take the points A and B at equal distances from D; then from the points A and B as centers, with a radius greater than $A\,D$, describe two arcs cutting each other at C; through C and D draw the straight line $C\,D$; it will be the perpendicular required.

For the point C, being equally distant from A and B, must be in a line perpendicular to the middle of $A\,B$ (Theo. X. Cor. 2, Bk. I.); hence $C\,D$ has been drawn perpendicular to $A\,B$ at the point D.

Scholium. The same construction serves for making a right angle, $A\,D\,C$, at a given point, D, on a given straight line, $A\,B$.

Problem IV. *To erect a perpendicular at the end of a given straight line.*

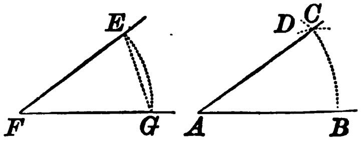
Let $A\,B$ be the straight line, and B the end of it at which a perpendicular is to be erected.



From any point, D, taken without the line $A\,B$, with a radius equal to the distance $D\,B$, describe an arc cutting the line *A B* at the points *A* and *B;* through the point *A*, and the center *D*, draw the diameter *A C*. Then through *C*, where the diameter meets the arc, draw the straight line *C B*, and it will be the perpendicular required.

For the angle *A B C*, being inscribed in a semicircle, is a right angle (Theo. X. Cor. 2, Bk. III.).

PROBLEM V. *At a point in a given straight line to make an angle equal to a given angle*

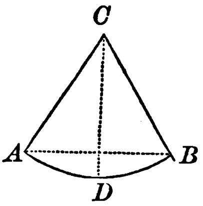


Let *A* be the given point, *A B* the given line, and *E F G* the given angle.

From the point *F* as a center, with any radius, describe an arc, *G E*, terminating in the sides of the angle; from the point *A* as a center, with the same radius, describe the indefinite arc *B D*. Draw the chord *G E;* then from *B* as a center, with a radius equal to *G E*, describe an arc cutting the arc *B D* in *C*. Draw *A C*, and the angle *C A B* will be equal to the given angle *E F G*.

For the two arcs, *B C* and *G E*, have equal radii and equal chords; therefore they are equal (Theo. II. Bk. III.); hence the angles *C A B*, *E F G*, measured by these arcs, are also equal (Theo. IV. Bk. III.).

PROBLEM VI. *To bisect a given arc, or a given angle.*



First. Let *A D B* be the given arc which it is required to bisect.

Draw the chord *A B;* from the center *C* draw the line *C D* perpendicular to *A B* (Prob. III.); it will bisect the arc *A D B* in the point *D*.

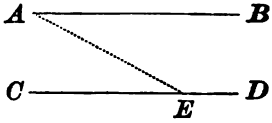
For *C D* being a radius perpendicular to a chord *A B*, must bisect the arc *A D B* which is subtended by that chord (Theo. V. Bk. III.).

Secondly. Let *A C B* be the angle which it is required to bisect. From *C* as a center, with any radius, describe the arc *A D B;* bisect this arc, as in the first case, by drawing the line *C D;* and this line will also bisect the angle *A C B*.

For the angles *A C D*, *B C D* are equal, being measured by the equal arcs *A D*, *D B* (Theo. IV. Bk. III.).

Scholium. By the same construction, we may bisect each of the halves *A D*, *D B;* and thus, by successive subdivisions, a given angle or a given arc may be divided into four equal parts, into eight, into sixteen, &c.

PROBLEM VII. *Through a given point, to draw a straight line parallel to a given straight line.*

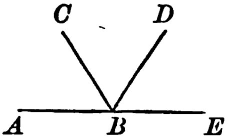


Let *A* be the given point, and *C D* the given straight line.

From *A* draw a straight line, *A E*, to any point, *E*, in *C D*. Then draw *A B*, making the angle *E A B* equal to the angle *A E C* (Prob. V.); and *A B* is parallel to *C D*.

For the alternate angles *E A B*, *A E C*, made by the straight line *A E* meeting the two straight lines *A B*, *C D*, being equal, the lines *A B* and *C D* must be parallel (Theo. XV. Bk. I.).

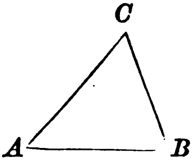
PROBLEM VIII. *Two angles of a triangle being given, to find the third angle.*



Draw the indefinite straight line *A B E*. At any point, *B*, make the angle *A B C* equal to one of the given angles (Prob. V.), and the angle *C B D* equal to the other given angle; then the angle *D B E* will be the third angle required.

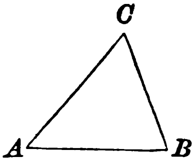
For these three angles are together equal to two right angles (Theo. I. Cor. 2, Bk. I.), as are also the three angles of every triangle (Theo. XIX. Bk. I.); and two of the angles at *B* having been made equal to two angles of the triangle, the remaining angle *D B E* must be equal to the third angle.

Problem IX. *Two sides of a triangle and the included angle being given, to construct the triangle.*



Draw the straight line *A B* equal to one of the two given sides. At the point *A* make an angle, *C A B*, equal to the given angle (Prob. V.); and take *A C* equal to the other given side. Join *B C*, and the triangle *A B C* will be the one required (Theo. IV. Bk. I.).

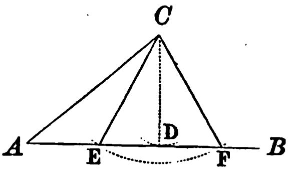
Problem X. *One side and two angles of a triangle being given, to construct the triangle.*



The two given angles will either be both adjacent to the given side, or one adjacent and the other opposite. In the latter case, find the third angle (Prob. VIII.); and the two angles adjacent to the given side will then be known.

In the former case, draw the straight line *A B* equal to the given side; at the point *A*, make an angle, *B A C*, equal to one of the adjacent angles, and at *B* an angle, *A B C*, equal to the other. Then the two sides *A C*, *B C* will meet, and form with *A B* the triangle required (Theo. V. Bk. I.).

Problem XI. *Two sides of a triangle and an angle opposite one of them being given, to construct the triangle.*



Draw the indefinite straight line *A B*. At the point *A* make an angle *B A C* equal to the given angle, and make *A C* equal to that side which is adjacent to the given angle. Then from *C*, as a center, with a radius equal to the other side, describe an arc, which must either touch the line *A B* in *D*, or cut it in the points *E* and *F*, otherwise a triangle could not be formed.

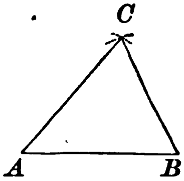
When the arc touches *A B*, a straight line drawn from *C* to the point of contact, *D*, will be perpendicular to *A B* (Theo.

VII. Cor. 1, Bk. III.), and the right-angled triangle *C A D* will be the triangle required.

When the arc cuts *A B* in two points, *E* and *F*, lying on the same side of the point *A*, draw the straight lines *C E*, *C F;* and each of the two triangles *C A E*, *C A F* will satisfy the conditions of the problem. If, however, the two points *E* and *F* should lie on different sides of the point *A*, only one of the triangles, as *C A F*, will satisfy all the conditions; hence that will be the triangle required.

Scholium. The problem would be impossible, if the side opposite the given angle were less than the perpendicular let fall from the point *C* on the straight line *A B*.

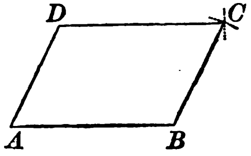
PROBLEM XII. *The three sides of a triangle being given, to construct the triangle.*



Draw the straight line *A B* equal to one of the given sides; from the point *A* as a center, with a radius equal to either of the other two sides, describe an arc; from the point *B*, with a radius equal to the third side, describe another arc cutting the former in the point *C;* draw the straight lines *A C*, *B C;* and the triangle *A B C* will be the one required (Theo. XIII. Bk. I.).

Scholium. The problem would be impossible, if one of the given sides were equal to or greater than the sum of the other two.

PROBLEM XIII. *Two adjacent sides of a parallelogram and the included angle being given, to construct the parallelogram.*

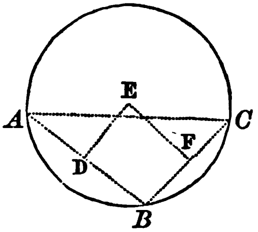


Draw the straight line *A B* equal to one of the given sides. At the point *A* make an angle, *B A D*, equal to the given angle, and take *A D* equal to the other given side. From the point D, with a radius equal to $A\,B$, describe an arc; and from the point B as a center, with a radius equal to $A\,D$, describe another arc cutting the former in the point C. Draw the straight lines $C\,D$, $C\,B$; and the parallelogram $A\,B\,C\,D$ will be the one required.

For the opposite sides are equal, by construction; hence the figure is a parallelogram (Theo. XXII. Bk. I.); and it is formed with the given sides and the given angle.

Cor. If the given angle is a right angle, the figure will be a rectangle; and if the adjacent sides are also equal, the figure will be a square.

PROBLEM XIV. *A circumference, or an arc, being given, to find the center of the circle.*



Take any three points, A, B, C, on the given circumference, or arc. Draw the chords $A\,B$, $B\,C$, and bisect them by the perpendiculars $D\,E$ and $F\,E$ (Prob. I.); the point E, in which these perpendiculars meet, is the center required.

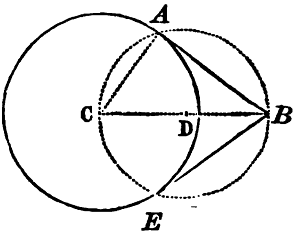
For the perpendiculars $D\,E$, $F\,E$ must both pass through the center (Theo. V. Cor. 2, Bk. III.), and E being the only point through which they both pass, E must be the center.

Scholium. By the same construction, a circumference may be made to pass through three given points, A, E, C, not in the same straight line; and also a circumference described in which a given triangle, $A\,B\,C$, shall be inscribed.

PROBLEM XV. *Through a given point to draw a tangent to a given circle.*

First. Let the given point A be in the circumference.

Find the center of the circle, C (Prob. XIV.); draw the radius $C\,A$; through the point A draw $A\,B$ perpendicular to $C\,A$ (Prob. IV.); and $A\,B$ will be the tangent required (Theo. VII. Bk. III.).



Secondly. Let the given point B be without the circumference.

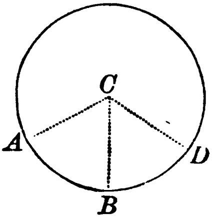
Join the point B and the center C by the straight line $B\,C$; bisect $B\,C$ in D; and from D as a center, with a radius equal to $C\,D$ or $D\,B$, describe a circumference intersecting the given circumference in the points A and E. Draw $A\,B$ and $E\,B$, and each will be a tangent as required.

For, drawing $C\,A$, the angle $C\,A\,B$, being inscribed in a semicircle, is a right angle (Theo. X. Cor. 2, Bk. III.); therefore $A\,B$ is perpendicular to the radius $C\,A$ at its extremity, A, and consequently is a tangent (Theo. VII. Bk. III.). In like manner it may be shown that $E\,B$ is a tangent.

EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

170. 1. The diameter is the longest straight line that can be inscribed in a circle.



2. A straight line cannot meet the circumference of a circle in more than two points.

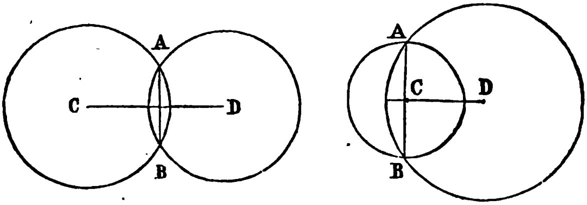
3. Two parallel tangents meet the circumference at the extremity of the same diameter.

4. If two straight lines intercept equal arcs of a circle, and do not cut each other within the circle, the lines will be parallel.

5. If a straight line be drawn to touch a circle, and be parallel to a chord, the point of contact will be the middle point of the arc cut off by that chord.

6. If two circumferences cut each other, the straight line

passing through their centers will bisect at right angles the chord which joins the points of intersection.



7. At a point in a given straight line to make an angle equal to half a right angle.

8. To describe a square whose side shall be equal to a given straight line.

BOOK IV.

AREAS AND RELATIONS OF PLANE FIGURES.

DEFINITIONS.

171. The **Area** of a figure is its quantity of surface, and is expressed by the number of times which the surface contains some other area assumed as a unit of measure.

Figures have *equal* areas, when they contain the same unit of measure an equal number of times.

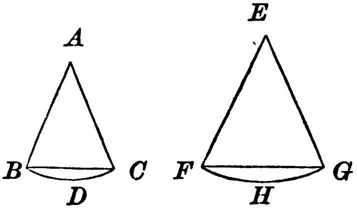
172. **Similar Figures** are such as have the angles of the one equal to those of the other, each to each, and the sides containing the equal angles proportional.

173. **Equivalent Figures** are such as have equal areas.

Figures may be equivalent which are not similar. Thus a circle may be equivalent to a square, and a triangle to a rectangle.

174. **Equal Figures** are such as, when applied the one to the other, coincide throughout (26, Ax. 12). Thus circles having equal radii are equal; and triangles having the three sides of the one equal to the three sides of the other, each to each, are also equal.

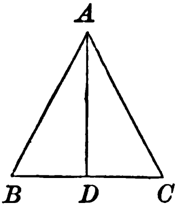
Equal figures are always similar; but similar figures may be very unequal.



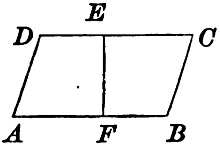
175. In different circles, **Similar Arcs, Segments.** or **Sectors** are such as correspond to equal angles at the centers of the circles.

Thus, if the angles A and E are equal, the arc $B\,C$ will be similar to the arc $F\,G$; the segment $B\,D\,C$

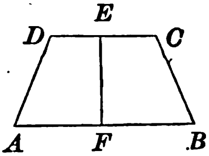
to the segment $F H G$, and the sector $A B C$ to the sector $E F G$.



176. **The Altitude of a Triangle** is the perpendicular, which measures the distance of any one of its vertices from the opposite side taken as a base; as the perpendicular $A D$ let fall on the base $B C$ in the triangle $A B C$.



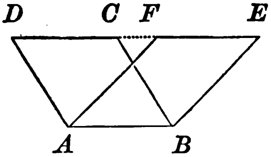
177. **The Altitude of a Parallelogram** is the perpendicular which measures the distance between its opposite sides taken as bases; as the perpendicular $E F$ measuring the distance between the opposite sides, $A B$, $D C$, of the parallelogram $A B C D$.



178. **The Altitude of a Trapezoid** is the perpendicular distance between its parallel sides; as the distance measured by the perpendicular $E F$ between the parallel sides, $A B$, $D C$, of the trapezoid $A B C D$.

THEOREM I.

179. *Parallelograms which have equal bases and equal altitudes are equivalent.*



Let $A B C D$, $A B E F$ be two parallelograms having equal bases and equal altitudes; then these parallelograms are equivalent.

Let the base of the one parallelogram be placed on that of the other, so that $A B$ shall be the common base. Now, since the two parallelograms are of the same altitude, their upper bases, $D C$, $F E$, will be in the same straight line, $D C E F$, parallel to $A B$. From the nature of parallelograms $D C$ is equal to $A B$, and $F E$ is equal to *A B* (Theo. XXII. Bk. I.); therefore *D C* is equal to *F E* (26, Ax. 1); hence if *D C* and *F E* be taken away from the same line, *D E*, the remainders *C E* and *D F* will be equal (26, Ax. 3). But *A D* is equal to *B C* and *A F* to *B E* (Theo. XXII. Bk. I.); therefore the triangles *D A F*, *C B E*, are mutually equilateral, and consequently equal (Theo. XIII. Bk. I.).

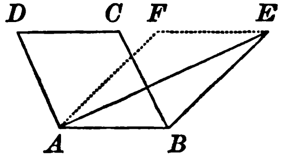
If from the quadrilateral *A B E D*, we take away the triangle *A D F*, there will remain the parallelogram *A B E F*; and if from the same quadrilateral *A B E D*, we take away the triangle *C B E*, there will remain the parallelogram *A B C D*. Hence the parallelograms *A B C D*, *A B E F*, which have equal bases and equal altitude, are equivalent.

180. *Cor.* Any parallelogram is equivalent to a rectangle having the same base and altitude.

THEOREM II.

181. *If a triangle and a parallelogram have the same base and altitude, the triangle is equivalent to half the parallelogram.*

Let *A B E* be a triangle, and *A B C D* a parallelogram having the same base, *A B*, and the same altitude; then will the triangle be equivalent to half the parallelogram.



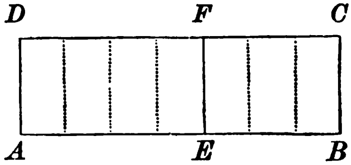
Draw *A F*, *F E* so as to form the parallelogram *A B E F*. Then the parallelograms *A B C D*, *A B E F*, having the same base and altitude, are equivalent (Theo. I.). But the triangle *A B E* is half the parallelogram *A B E F* (Theo. XXII. Cor. 1, Bk. I.); hence the triangle *A B E* is equivalent to half the parallelogram *A B C D* (26, Ax. 7).

182. *Cor.* 1. Any triangle is equivalent to half a rectangle having the same base and altitude, or to a rectangle either having the same base and half of the same altitude, or having the same altitude and half of the same base.

183. *Cor.* 2. All triangles which have equal bases and altitudes are equivalent.

THEOREM III.

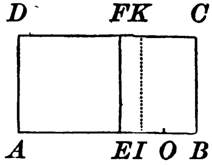
184. *Two rectangles having equal altitudes are to each other as their bases.*



Let $ABCD$, $AEFD$ be two rectangles having the common altitude AD; they are to each other as their bases AB, AE.

First. Suppose that the bases AB, AE are commensurable, and are to each other, for example, as the numbers 7 and 4. If AB is divided into seven equal parts, AE will contain four of those parts. At each point of division draw lines perpendicular to the base; seven rectangles will thus be formed, all equal to each other, since they have equal bases and the same altitude (Theo. I.) The rectangle $ABCD$ will contain seven partial rectangles, while $AEFD$ will contain four; hence the rectangle $ABCD$ is to $AEFD$ as 7 is to 4, or as AB is to AE. The same reasoning may be applied, whatever be the numbers expressing the ratio of the bases; hence, whatever be that ratio, when its terms are commensurable, we shall have

$$ABCD : AEFD :: AB : AE.$$



Second. Suppose that the bases AB, AE are incommensurable; we shall still have

$$ABCD : AEFD :: AB : AE.$$

For, if this proportion be not true, the first three terms remaining the same, the fourth term must be either greater or less than $A\,E$. Suppose it to be greater, and that we have

$$ABCD : AEFD :: AB : AO.$$

Conceive $A\,B$ divided into equal parts, each of which is less than $E\,O$. There will be at least one point of division, I, between E and O. Through this point, I, draw the perpendicular $I\,K$; then the bases $A\,B$, $A\,I$ will be commensurable, and we shall have

$$ABCD : AIKD :: AB : AI.$$

But, by hypothesis, we have

$$ABCD : AEFD :: AB : AO.$$

In these two proportions the antecedents are equal; hence the consequents are proportional (Theo. IX. Cor. 2, Bk. II.), and we have

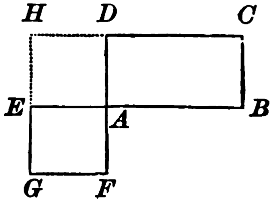
$$AIKD : AEFD :: AI : AO.$$

But $A\,O$ is greater than $A\,I$; therefore, if this proportion is correct, the rectangle $A\,E\,F\,D$ must be greater than the rectangle $A\,I\,K\,D$ (94); on the contrary, however, it is less (26, Ax. 8); therefore the proportion is impossible. Hence, $A\,B\,C\,D$ cannot be to $A\,E\,F\,D$ as $A\,B$ is to a line greater than $A\,E$.

In the same manner, it may be shown that the fourth term of the proportion cannot be less than $A\,E$; therefore it must be equal to $A\,E$. Hence, any two rectangles $A\,B\,C\,D$, $A\,E\,F\,D$, having equal altitudes, are to each other as their bases $A\,B$, $A\,E$.

Theorem IV.

185. *Any two rectangles are to each other as the products of their bases multiplied by their altitudes.*



Let $A\,B\,C\,D$, $A\,E\,G\,F$ be two rectangles; then will $A\,B\,C\,D$ be to $A\,E\,G\,F$ as $A\,B$ multiplied by $A\,D$ is to $A\,E$ multiplied by $A\,F$. Having placed the two rectangles so that the angles at A are vertical, produce the sides $G\,E$, $C\,D$ till they meet in H. The two rectangles $A\,B\,C\,D$, $A\,E\,H\,D$, having the same altitude $A\,D$, are to each other as their bases, $A\,B$, $A\,E$. In like manner the two rectangles $A\,E\,H\,D$, $A\,E\,G\,F$, having the same altitude, $A\,E$, are to each other as their bases, $A\,D$, $A\,F$. Hence we have the two proportions,

$$A\,B\,C\,D : A\,E\,H\,D :: A\,B : A\,E,$$
$$A\,E\,H\,D : A\,E\,G\,F :: A\,D : A\,F.$$

Multiplying the corresponding terms of these proportions together (Theo. XI. Bk. II.), and omitting the factor $A\,E\,H\,D$, which is common to both the antecedent and the consequent (Theo. VIII. Bk. II.), we shall have

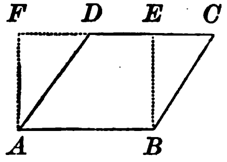
$$A\,B\,C\,D : A\,E\,G\,F :: A\,B \times A\,D : A\,E \times A\,F.$$

186. *Scholium.* Hence, we may assume as the measure of a rectangle, the product of its base by its altitude, provided we understand by this product the product of two numbers, one of which represents the number of linear units contained in the base, the other the number of linear units contained in the altitude.

The product of two lines is often used to designate their *rectangle;* but the term *square* is used to designate the product of a number multiplied by itself.

THEOREM V.

187. *The area of any parallelogram is equal to the product of its base by its altitude.*



Let $A\,B\,C\,D$ be any parallelogram, $A\,B$ its base, and $E\,B$ its altitude; then will its area be equal to the product of $A\,B$ by $B\,E$.

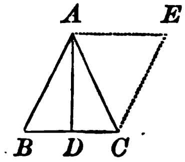
Draw $B\,E$ and $A\,F$ perpendicular to $A\,B$, and produce $C\,D$ to F. Then the parallelogram $A\,B\,C\,D$ is equivalent to the rectangle $A\,B\,E\,F$, which has the same base, $A\,B$, and the same altitude, $B\,E$ (Theo. I.

Cor.). But the rectangle $ABEF$ is measured by $AB \times BE$ (Theo. IV. Sch.); therefore $AB \times BE$ is equal to the area of the parallelogram $ABCD$.

188. *Cor.* Parallelograms having equal bases are to each other as their altitudes, and parallelograms having equal altitudes are to each other as their bases; and, in general, parallelograms are to each other as the products of their bases by their altitudes.

THEOREM VI.

189. *The area of any triangle is equal to the product of its base by half its altitude.*



Let ABC be any triangle, BC its base, and AD its altitude; then its area will be equal to the product of BC by half of AD.

Draw AE and CE so as to form the parallelogram $ABCE$; then the triangle ABC is half the parallelogram $ABCE$, which has the same base BC, and the same altitude AD (Theo. II.); but the area of the parallelogram is equal to $BC \times AD$ (Theo. V.); hence the area of the triangle must be $\frac{1}{2} BC \times AD$, or $BC \times \frac{1}{2} AD$.

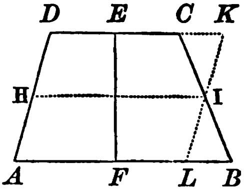
190. *Cor.* Triangles of equal altitudes are to each other as their bases, and triangles of equal bases are to each other as their altitudes; and, in general, triangles are to each other as the products of their bases and altitudes.

THEOREM VII.

191. *The area of any trapezoid is equal to the product of its altitude by half the sum of its parallel sides.*

Let $ABCD$ be a trapezoid, EF its altitude, and AB, CD its parallel sides; then its area will be equal to the product EF by half the sum of AB and CD.

Through I, the middle point of the side BC, draw KL



parallel to *A D*; and produce *D C* till it meet *K L*. In the triangles *I B L*, *I C K*, we have the sides *I B*, *I C* equal by construction; the vertical angles *L I B*, *C I K* are equal (Theo. III. Bk. I.); and, since *C K* and *B L* are parallel, the alternate angles *I B L*, *I C K* are also equal (Theo. XVIII. Bk. I.); therefore the triangles *I B L*, *I C K* are equal (Theo. V. Bk. I.); hence the trapezoid *A B C D* is equivalent to the parallelogram *A D K L*, and is measured by the product of *E F* by *A L* (Theo. V.).

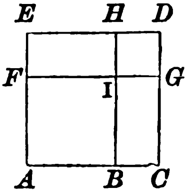
But we have *A L* equal *D K*; and since the triangles *I B L* and *K C I* are equal, the sides *B L* and *C K* are equal; therefore the sum of *A B* and *C D* is equal to the sum of *A L* and *D K*, or twice *A L*. Hence *A L* is half the sum of the bases *A B*, *C D*; hence the area of the trapezoid *A B*, *C D* is equal to the product of the altitude *E F* by half the sum of the parallel sides *A B*, *C D*.

Cor. If through *I*, the middle point of *B C*, the line *I H* be drawn parallel to the base *A B*, the point *H* will also be the middle point of *A D*. For, since the figure *A H I L* is a parallelogram, as is likewise *D H I K*, their opposite sides being parallel, we have *A H* equal to *I L*, and *D H* equal to *I K*. But since the triangles *B I L*, *C I K* are equal, we have *I L* equal to *I K*; hence *A H* is equal to *D H*.

Now, the line *H I* is equal to *A L*, which has been shown to be equal to half the sum of *A B* and *C D*; therefore the area of the trapezoid is equal to the product of *E F* by *H I*. Hence, the area of a trapezoid is equal to the product of its altitude by the line connecting the middle points of the sides which are not parallel.

THEOREM VIII.

192. *If a straight line be divided into two parts, the square described on the whole line is equivalent to the sum* *of the squares described on the parts, together with twice the rectangle contained by the parts.*

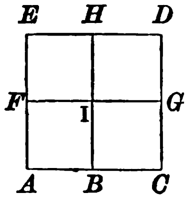


Let $A\,C$ be a straight line, divided into two parts, $A\,B$, $B\,C$, at the point B; then the square described on $A\,C$ is equivalent to the sum of the squares described on the parts $A\,B$, $B\,C$, together with twice the rectangle contained by $A\,B$, $B\,C$; that is,

$$\overline{A\,C}^2 = \overline{A\,B}^2 + \overline{B\,C}^2 + 2\,A\,B \times B\,C.$$

On $A\,C$ describe the square $A\,C\,D\,E$; take $A\,F$ equal to $A\,B$; draw $F\,G$ parallel to $A\,C$, and $B\,H$ parallel to $A\,E$.

The square $A\,C\,D\,E$ is divided into four parts; the first, $A\,B\,I\,F$, is the square described on $A\,B$, since $A\,F$ was taken equal to $A\,B$. The second, $I\,G\,D\,H$, is the square described upon $B\,C$; for, since $A\,C$ is equal to $A\,E$, and $A\,B$ is equal to $A\,F$, $A\,C$ minus $A\,B$ is equal to $A\,E$ minus $A\,F$, which gives $B\,C$ equal to $E\,F$. But $I\,G$ is equal to $B\,C$, and $D\,G$ to $E\,F$, since the lines are parallels; therefore $I\,G\,D\,H$ is equal to the square described on $B\,C$. These two parts being taken from the whole square, there remain two rectangles $B\,C\,G\,I$, $E\,F\,I\,H$, each of which is measured by $A\,B \times B\,C$; hence the square on the whole line $A\,C$ is equivalent to the squares on the parts $A\,B$, $B\,C$, together with twice the rectangle of the parts.



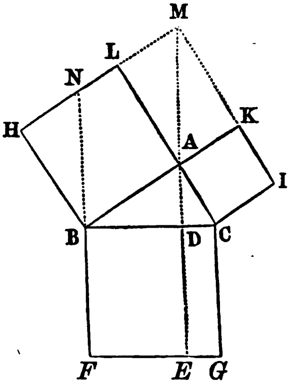
193. *Cor.* The square described on the whole line $A\,C$ is equivalent to four times the square described on the half $A\,B$.

194. *Scholium.* This proposition is equivalent to the algebraical formula,

$$(a+b)^2 = a^2 + 2\,a\,b + b^2.$$

THEOREM IX.

195. *The square described on the hypothenuse of a right-angled triangle is equivalent to the sum of the squares described on the other two sides.*



Let $A B C$ be a right-angled triangle, having the right angle at A; then the square described on the hypothenuse $B C$ will be equivalent to the sum of the squares on the sides $B A$, $A C$.

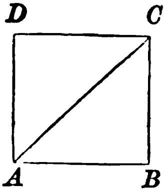
On $B C$ describe the square $B C G F$, and on $A B$, $A C$ the squares $A B H L$, $A C I K$; produce $F B$ to N, $H L$ and $I K$ to M; and through A draw $E D A$ parallel to $F B N$, and meeting the prolongation of $H L$ in M.

Then, since the angles $H B A$, $N B C$ are both right angles, if the common angle $N B A$ be taken from each of these equals, there will remain the equal angles $H B N$, $A B C$; and, consequently, since the triangles $H B N$, $A B C$ are both right-angled, and have also the sides $B H$, $B A$ equal, their hypothenuses $B N$, $B C$ are equal (Theo. V. Cor., Bk. I.). But $B C$ is equal to $B F$; therefore $B N$ is equal to $B F$; hence the parallelograms $B A M N$, $B D E F$, of which the common altitude is $B D$, have equal bases; therefore the two parallelograms are equivalent (Theo. I.). But the parallelogram $B A M N$ is equivalent to the square $A B H L$, since they have the same base $B A$, and the same altitude $A L$; hence the parallelogram $B D E F$ is also equivalent to the square $A B H L$. In like manner it may be shown that the rectangle $D C G E$ is equivalent to the square $A C I K$; hence the two rectangles together, that is, the square $B C G F$, are equivalent to the sum of the squares $A B H L$, $A C I K$, or

$$\overline{B C}^2 \text{ is equivalent to } \overline{A B}^2 + \overline{A C}^2.$$

196. *Cor.* 1. *The square of either of the sides which form the right angle of a right-angled triangle is equivalent to the square of the hypothenuse diminished by the square of the other side;* thus,

$$\overline{AB}^2 \text{ is equivalent to } \overline{BC}^2 - \overline{AC}^2.$$



197. *Cor.* 2. *The square described on the diagonal of a square is equivalent to double the square described on a side.* For let $ABCD$ be a square, and AC its diagonal; the triangle ABC being right-angled and isosceles, we have,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = 2\,\overline{AB}^2 = 2 \times ABCD.$$

198. *Cor.* 3. Since $\overline{AC}^2$ is equal to $2\,\overline{AB}^2$, we have

$$\overline{AC}^2 : \overline{AB}^2 :: 2 : 1;$$

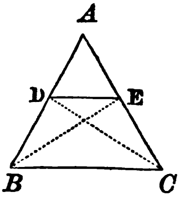
and, extracting the square root, we have

$$AC : AB :: \sqrt{2} : 1;$$

hence, *the diagonal of a square is incommensurable with a side.*

THEOREM X.

199. *If a straight line be drawn in a triangle parallel to one of the sides, it will divide the other two sides proportionally.*



Let ABC be a triangle, and DE a straight line drawn within it parallel to the side BC; then will

$$AD : DB :: AE : EC.$$

Join BE and DC; then the two triangles BDE, DEC have the same base, DE; they also have the same altitude, since the vertices B and C lie in a line parallel to the base; therefore the triangles are equivalent (Theo. II. Cor. 2).

The triangles ADE, BDE, having their bases in the same line $A B$, and having the common vertex E, have the same altitude, and therefore are to each other as their bases (Theo. VI. Cor.); hence

$$ADE : BDE :: AD : DB.$$

The triangles $A D E$, $D E C$, whose common vertex is D, have also the same altitude, and therefore are to each other as their bases; hence

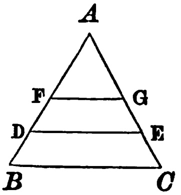
$$ADE : DEC :: AE : EC.$$

But the triangles $B D E$, $D E C$ have been shown to be equivalent; therefore, on account of the common ratio in the two proportions (Theo. IX. Bk. II.),

$$AD : DB :: AE : EC.$$

200. *Cor.* 1. Hence, by composition (Theo. VI. Bk. II.), we have $AD + DB : AD :: AE + EC : AE$; or $AB : AD :: AC : AE$; also, $AB : BD :: AC : EC$.

201. *Cor.* 2. If two or more straight lines be drawn in a triangle parallel to one of the sides, they will divide the other two sides proportionally.

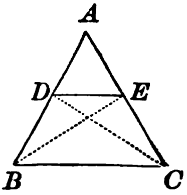


For, in the triangle $A B C$, since $D E$ is parallel to $B C$, by the theorem, $AD : DB :: AE : EC$; and, in the triangle $A D E$, since $F G$ is parallel to $D E$, by the preceding corollary, $AD : FD :: AE : GE$. Hence, since the antecedents are the same in the two proportions (Theo. IX. Cor. 2, Bk. II.), $FD : DB :: GE : EC$.

THEOREM XI.

202. *If a straight line divides two sides of a triangle proportionally, the line is parallel to the other side of the triangle.*

Let $A B C$ be a triangle, and $D E$ a straight line drawn in it dividing the sides $A B$, $A C$, so that $AD : DB :: AE : EC$; then will the line $D E$ be parallel to the side $B C$.



Join BE and DC; then the triangles ADE, BDE, having their bases in the same straight line AB, and having a common vertex, E, are to each other as their bases AD, DB (Theo. VI. Cor.); that is,

$$ADE : BDE :: AD : DB.$$

Also, the triangles ADE, DEC, having the common vertex D, and their bases in the same line, are to each other as these bases, AE, EC; that is,

$$ADE : DEC :: AE : EC.$$

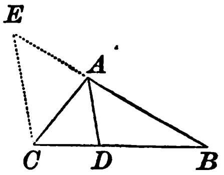
But, by hypothesis, $AD : DB :: AE : EC$; hence (Bk. II.),

$$ADE : BDE :: ADE : DEC;$$

that is, BDE, DEC have the same ratio to ADE; therefore the triangles BDE, DEC have the same area, and consequently are equivalent (173). Since these triangles have the same base, DE, their altitudes are equal (Theo. VI. Cor.); hence the line BC, in which their vertices are, must be parallel to DE.

THEOREM XII.

203. *The straight line bisecting any angle of a triangle divides the opposite side into parts, which are proportional to the adjacent sides.*



In any triangle, ABC, let the angle BAC be bisected by the straight line AD; then will

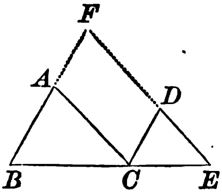
$$BD : DC :: AB : AC.$$

Through the point C draw CE parallel to AD, meeting BA produced in E. Then, since the two parallels AD, EC are met by the straight line AC, the alternate angles DAC, ACE are equal (Theo. XVII. Bk. I.); and the same parallels being met by the straight line BE, the opposite exterior and interior angles BAD, AEC are also equal (Theo. XVII. Bk. I.). But, by hypothesis, the angles DAC, BAD are equal; consequently the angle ACE is equal to the angle AEC; hence the triangle ACE is isosceles, and the side AE is equal to the side AC (Theo. VII. Bk. I.). Again, since AD, in the triangle EBC, is parallel to EC, we have $BD : DC :: AB : AE$ (Theorem X.), and, substituting AC in place of its equal AE,

$$BD : DC :: AB : AC.$$

THEOREM XIII.

204. *Equiangular triangles have their homologous sides proportional, and are similar.*



Let the two triangles ABC, DCE be equiangular; the angle BAC being equal to the angle CDE, the angle ABC to the angle DCE, and the angle ACB to the angle DEC, then the homologous sides will be proportional, and we shall have

$$BC : CE :: AB : CD :: AC : DE.$$

For, let the two triangles be placed so that two homologous sides, BC, CE, may join each other, and be in the same straight line; and produce the sides BA, ED till they meet in F.

Since BCE is a straight line, and the angle BCA is equal to the angle CED, AC is parallel to FE (Theo. XVI. Bk. I.); also, since the angle ABC is equal to the angle DCE, the line BF is parallel to the line CD. Hence the figure $ACDF$ is a parallelogram; and, consequently, AF is equal to CD, and AC to FD (Theo. XXII. Bk. I.).

In the triangle BEF, since the line AC is parallel to the side $F E$, we have $B C : C E :: B A : A F$ (Theo. X.); or, substituting $C D$ for its equal, $A F$,

$$B C : C E :: B A : C D.$$

Again, $C D$ is parallel to $B F$; therefore, $B C : C E :: F D : D E$; or, substituting $A C$ for its equal $F D$,

$$B C : C E :: A C : D E.$$

And, since both these proportions contain the same ratio $B C : C E$, we have (Theo. IX. Bk. II.)

$$A C : D E :: B A : C D.$$

Hence, the equiangular triangles $B A C$, $C D E$ have their homologous sides proportional; and consequently the two triangles are similar (172).

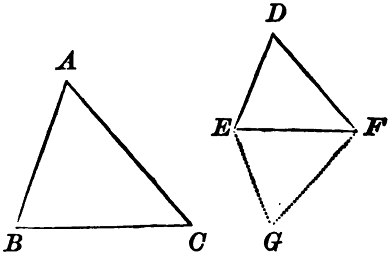
205. *Cor.* Two triangles having two angles of the one equal to two angles of the other, each to each, are similar; since the third angles will also be equal, and the two triangles be equiangular.

206. *Scholium.* In similar triangles, the homologous sides are opposite to the equal angles; thus the angle $A C B$ being equal to $D E C$, the side $A B$ is homologous to $D C$; in like manner, $A C$ and $D E$ are homologous.

Theorem XIV.

207. *Triangles which have their homologous sides proportional, are equiangular and similar.*

Let the two triangles $A B C$, $D E F$ have their sides proportional, so that we have $B C : E F :: A B : D E :: A C : D F$; then will the triangles have their angles equal; namely, the angle A equal to the angle D, the angle B to the angle E, and the angle C to the angle F.



At the point E, in the straight line EF, make the angle FEG equal to the angle B, and at the point F, the angle EFG equal to the angle C; the third angle G will be equal to the third angle A (Theo. XIX. Cor. 2, Bk. I.); and the two triangles ABC, EFG will be equiangular. Therefore, by the last theorem, we have

$$BC : EF :: AB : EG;$$

but, by hypothesis, we have

$$BC : EF :: AB : DE;$$

hence, EG is equal to DE.

By the same theorem, we also have

$$BC : EF :: AC : FG;$$

and, by hypothesis,

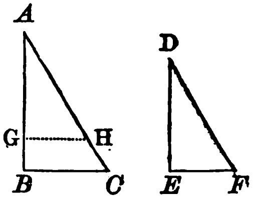
$$BC : EF :: AC : DF;$$

hence FG is equal to DF. Hence, the triangles EGF, DEF, having their three sides equal, each to each, are themselves equal (Theo. XIII. Bk. I.). But, by construction, the triangle EGF is equiangular with the triangle ABC; hence the triangles DEF, ABC are also equiangular and similar.

THEOREM XV.

208. *Two triangles, which have an angle of the one equal to an angle of the other, and the sides containing these angles proportional, are similar.*

Let the two triangles ABC, DEF have the angle A equal to the angle D, and the sides containing these angles proportional, so that $AB : DE :: AC : DF$; then the triangles are similar.



Take AG equal DE, and draw GH parallel to BC. The angle AGH will be equal to the angle ABC (Theo. XVII. Bk. I.); and the triangles AGH, ABC will be equiangular; hence we shall have

$$AB : AG :: AC : AH.$$

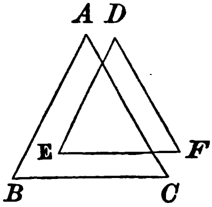
But, by hypothesis,

$$AB : DE :: AC : DF;$$

and, by construction, AG is equal to DE; hence AH is equal to DF. Therefore the two triangles AGH, DEF, having two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, are themselves equal (Theo. IV. Bk. I.) But the triangle AGH is similar to ABC; therefore DEF is also similar to ABC.

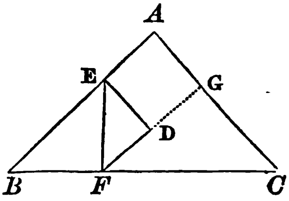
THEOREM XVI.

209. *Two triangles, which have their sides, taken two and two, either parallel or perpendicular to each other, are similar.*



Let the two triangles ABC, DEF have the side AB parallel to the side DE, BC parallel to EF, and AC parallel to DF; these triangles will then be similar.

For, since the side AB is parallel to the side DE, and BC to EF, the angle ABC is equal to the angle DEF (Theo. XVIII. Bk. I.). Also, since AC is parallel to DF, the angle ACB is equal to the angle DFE, and the angle BAC to EDF; therefore the triangles ABC, DEF are equiangular; hence they are similar (Theo. XIII.).



Again, let the two triangles ABC, DEF have the side DE perpendicular to the side AB, DF perpendicular to AC, and EF perpendicular to BC; these triangles are similar.

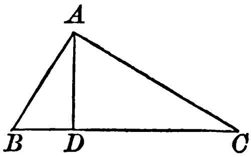
Produce *F D* till it meets *A C* at *G;* then the angles *D G A*, *D E A* of the quadrilateral *A E D G* are two right angles; and since all the four angles are together equal to four right angles (Theo. XXI. Cor. 1, Bk. I.), the remaining two angles, *E D G*, *E A G*, are together equal to two right angles. But the two angles *E D G*, *E D F* are also together equal to two right angles (Theo. I. Bk. I.); hence the angle *E D F* is equal to *E A G* or *B A C*.

The two angles, *G F C*, *G C F*, in the right-angled triangle *F G C*, are together equal to a right angle (Theo. XIX. Cor. 1, Bk. I.), and the two angles *G F C*, *G F E* are together equal to the right angle *E F C* (26, Ax. 9); therefore *G F E* is equal to *G C F*, or *D F E* to *B C A*. Therefore the triangles *A B C*, *D E F* have two angles of the one equal to two angles of the other, each to each; hence they are similar (Theo. XIII. Cor.)

Scholium. When the two triangles have their sides parallel, the parallel sides are homologous; and when they have them perpendicular, the perpendicular sides are homologous. Thus, *D E* is homologous with *A B*, *D F* with *A C*, and *E F* with *B C*.

THEOREM XVII.

210. *In a right-angled triangle, if a perpendicular is drawn from the vertex of the right angle to the hypothenuse, the triangle will be divided into two triangles similar to the given triangle and to each other.*



In the right-angled triangle *A B C*, from the vertex of the right angle *B A C*, let *A D* be drawn perpendicular to the hypothenuse *B C;* then the triangles *B A D*, *D A C* will be similar to the triangle *A B C*, and to each other.

For the triangles *B A D*, *B A C* have the common angle

B, the right angle *B D A* equal to the right angle *B A C*, and therefore the third angle, *B A D*, of the one, equal to the the third angle, *C*, of the other (Theo. XIX. Cor. 2, Bk. I.); hence these two triangles are equiangular, and consequently are similar (Theo. XIII.). In the same manner it may be shown that the triangles *D A C* and *B A C* are equiangular and similar. The triangles *B A D* and *D A C*, being each similar to the triangle *B A C*, are similar to each other.

211. *Cor.* 1. Each of the sides about a right angle is a mean proportional between the hypothenuse and the part of it which is cut off adjacent to that side by the perpendicular from the vertex of the right angle.

For, the triangles *B A D*, *B A C* being similar, their homologous sides are proportional; hence

$$BD : BA :: BA : BC;$$

and, the triangles *D A C*, *B A C* being also similar,

$$DC : AC :: AC : BC;$$

hence each of the sides *A B*, *A C* is a mean proportional between the hypothenuse and the part cut off adjacent to that side.

212. *Cor.* 2. The perpendicular from the vertex of the right angle to the hypothenuse is a mean proportional between the two parts into which it divides the hypothenuse.

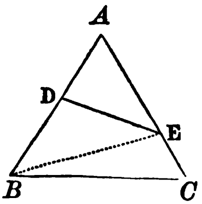
For, since the triangles *A B D*, *A D C* are similar, by comparing their homologous sides we have

$$BD : AD :: AD : DC;$$

hence, the perpendicular *A D* is a mean proportional between the parts *D B*, *D C* into which it divides the hypothenuse *B C*.

THEOREM XVIII.

213. *Two triangles, having an angle in each equal, are to each other as the rectangles of the sides which contain the equal angles.*



Let the two triangles ABC, ADE have the angle A in common; then will the triangle ABC be to the triangle ADE as $AB \times AC$ to $AD \times AE$.

Join BE; then the triangles ABE, ADE, having the common vertex E, and their bases in the same line, AB, have the same altitude, and are to each other as their bases (Theo. VI. Cor.); hence

$$ABE : ADE :: AB : AD.$$

In like manner, since the triangles ABC, ABE have the common vertex B, and their bases in the same line, AC, we have

$$ABC : ABE :: AC : AE.$$

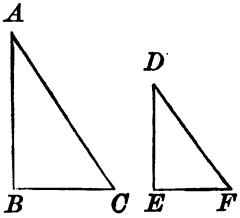
By multiplying together the corresponding terms of these proportions, and omitting the common term ABE, we have (Theo. XI. Bk. II.),

$$ABC : ADE :: AB \times AC : AD \times AE.$$

214. *Cor.* If the rectangles of the sides containing the equal angles were equivalent, the triangles would be equivalent.

Theorem XIX.

215. *Similar triangles are to each other as the squares described on their homologous sides.*



Let ABC, DEF be two similar triangles, and let AC, DF be homologous sides; then the triangle ABC will be to the triangle DEF as the square on AC is to the square on DF.

For, the triangles being similar, they have their homologous sides proportional (172); therefore

$$AB : DE :: AC : DF;$$

and multiplying the terms of this proportion by the corresponding terms of the identical proportion,

$$A\,C : D\,F :: A\,C : D\,F,$$

we have (Theo. XI. Bk. II.),

$$A\,B \times A\,C : D\,E \times D\,F :: \overline{A\,C}^2 : \overline{D\,F}^2.$$

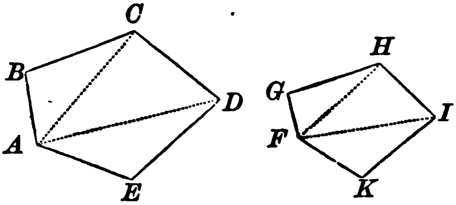
But, by reason of the equal angles A and D, the triangle $A\,B\,C$ is to the triangle $D\,E\,F$ as $A\,B \times A\,C$ is to $D\,E \times D\,F$ (Theo. XVIII.); consequently (Theo. IX. Bk. II.),

$$A\,B\,C : D\,E\,F :: \overline{A\,C}^2 : \overline{D\,F}^2.$$

Therefore, the two similar triangles $A\,B\,C$, $D\,E\,F$ are to each other as the squares described on the homologous sides $A\,C$, $D\,F$, or as the squares described on any other two homologous sides.

THEOREM XX.

216. *Similar polygons may be divided into the same number of triangles similar each to each, and similarly situated.*



Let $A\,B\,C\,D\,E$, $F\,G\,H\,I\,K$ be two similar polygons; they may be divided into the same number of triangles similar each to each, and similarly situated. From the homologous angles A and F, draw the diagonals $A\,C$, $A\,D$ and $F\,H$, $F\,I$.

The two polygons being similar, the angles B and G, which are homologous, must be equal, and the sides $A\,B$, $B\,C$ must also be proportional to $F\,G$, $G\,H$ (172); that is, $A\,B : F\,G :: B\,C : G\,H$. Therefore the triangles $A\,B\,C$, $F\,G\,H$ have an angle of the one equal to the angle of the other, and the sides containing these angles proportional; hence they are similar (Theo. XV.); consequently the angle $B\,C\,A$ is equal to the angle GHF. These equal angles being taken from the equal angles BCD, GHI, the remaining angles ACD, FHI will be equal (26, Ax. 3). But, since the triangles ABC, FGH are similar, we have

$$AC : FH :: BC : GH;$$

and, since the polygons are similar (172),

$$BC : GH :: CD : HI;$$

hence (Theo. IX. Cor. 1, Bk. II.),

$$AC : FH :: CD : HI.$$

But the terms of the last proportion are the sides about the equal angles ACD, FHI; hence the triangles ACD, FHI are similar (Theo. XV.). In the same manner, it may be shown that the corresponding triangles ADE, FIK are similar; hence the similar polygons may be divided into the same number of triangles similar each to each, and similarly situated.

217. *Cor. Conversely, if two polygons are composed of the same number of similar triangles, and similarly situated, the two polygons are similar.*

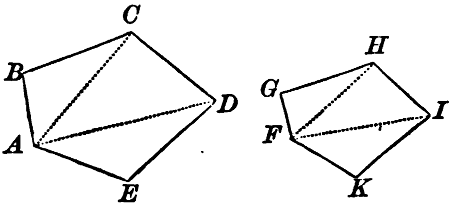
For the similarity of the corresponding triangles, give the angles ABC equal to FGH, BCA equal to GHF, and ACD equal to FHI; hence, BCD equal to GHI, likewise CDE equal to HIK, &c. Moreover, we have

$AB : FG :: BC : GH :: AC : FH :: CD : HI$, &c.; therefore the two polygons have their angles equal and their sides proportional; hence they are similar.

THEOREM XXI.

218. *The perimeters of similar polygons are to each other as their homologous sides; and their areas are to each other as the squares described on these sides.*

Let $ABCDE$, $FGHIK$ be two similar polygons; then their perimeters are to each other as their homologous sides AB and FG, BC and GH, &c.; and their areas are to each other as $\overline{AB}^2$ is to $\overline{FG}^2$, $\overline{BC}^2$ to $\overline{GH}^2$, &c.



First. Since the two polygons are similar, we have

$$AB : FG :: BC : GH :: CD : HI, \text{ \&c.}$$

Now the sum of the antecedents AB, BC, CD, &c., which compose the perimeter of the first polygon, is to the sum of the consequents FG, GH, HI, &c., which compose the perimeter of the second polygon, as any one antecedent is to its consequent (Theo. X. Bk. II.); therefore, as any two homologous sides are to each other, or as AB is to FG.

Secondly. From the homologous angles A and F, draw the diagonals AC, AD and FH, FI. Then, since the triangles ABC, FGH are similar, the triangle

$$ABC : FGH :: \overline{AC}^2 : \overline{FH}^2$$

(Theo. XIX.); and, since the triangles ACD, FHI are similar, the triangle $ACD : FHI :: \overline{AC}^2 : \overline{FH}^2$. But the ratio $\overline{AC}^2 : \overline{FH}^2$ is common to both of the proportions; therefore (Theo. IX. Bk. II.),

$$ABC : FGH :: ACD : FHI.$$

Bv the same mode of reasoning, it may be proved that

$$ACD : FHI :: ADE : FIK,$$

and so on, if there were more triangles. Therefore the sum of the antecedents ABC, ACD, ADE, which compose the area of the polygon $ABCDE$, is to the sum of the consequents FGH, FHI, FIK, which compose the area of the polygon $FGHIK$, as any one antecedent ABC is to its consequent FGH (Theo. X. Bk. II.), or as $\overline{AB}^2$ is to $\overline{FG}^2$;

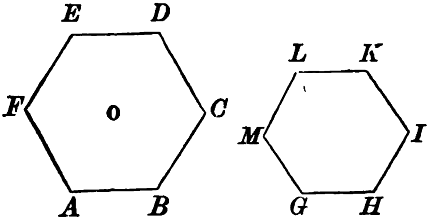
hence the areas of similar polygons are to each other as the squares described on their homologous sides.

219. *Cor.* 1. The perimeters of similar polygons are also to each other as their corresponding diagonals.

220. *Cor.* 2. The areas of similar polygons are to each other as the squares described on their corresponding diagonals.

THEOREM XXII.

221. *Regular polygons of the same number of sides are similar figures.*



Let $ABCDEF$, $GHIKLM$ be two regular polygons of the same number of sides; then these polygons are similar.

For, since the two polygons have the same number of sides, they have the same number of angles; and the sum of all the angles is the same in the one as in the other (Theo. XX. Bk. I.). Also, since the polygons are equiangular, each of the angles A, B, C, &c. is equal to each of the angles G, H, I, &c.; hence the two polygons are mutually equiangular.

Again; the polygons being regular, the sides AB, BC, CD, &c. are equal to each other; so likewise are the sides GH, HI, IK, &c. Hence,

$$AB : GH :: BC : HI :: CD : IK, \text{ \&c.}$$

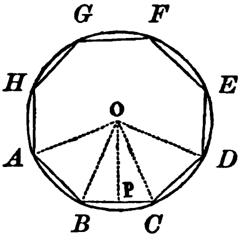
Therefore the two polygons have their angles equal, and their homologous sides proportional; hence they are similar (176).

222. *Cor.* The perimeters of two regular polygons of the same number of sides are to each other as their homologous sides, and their areas are to each other as the squares of those sides (Theo. XXI.).

223. *Scholium.* The angle of a regular polygon is determined by the number of its sides (Theo. XX. Bk. I.).

THEOREM XXIII.

224. *A circle may be circumscribed about, and another inscribed in, any regular polygon.*



Let $ABCDEFGH$ be any regular polygon; then a circle may be circumscribed about, and another inscribed in it.

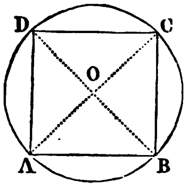
Describe a circle whose circumference shall pass through the three points A, B, C, the center being O; let fall the perpendicular OP from O to the middle point of the side BC; and draw the straight lines OA, OB, OC, OD.

Now, if the quadrilateral $OPCD$ be placed upon the quadrilateral $OPBA$, they will coincide; for the side OP is common, and the angle OPC is equal to the angle OPB, each being a right angle; consequently the side PC will fall upon its equal, PB, and the point C on B. Moreover, from the nature of the polygon, the angle PCD is equal to the angle PBA; therefore CD will take the direction BA, and CD being equal to BA, the point D will fall upon A, and the two quadrilaterals will coincide throughout. Therefore OD is equal to AO, and the circumference which passes through the three points A, B, C, will also pass through the point D. By the same mode of reasoning, it may be shown that the circle which passes through the three vertices B, C, D, will also pass through the vertex E, and so on. Hence, the circumference which passes through the three points A, B, C, passes through the vertices of all the *angles* of the polygon, and is circumscribed about the polygon (134).

Again, with respect to this circumference, all the sides, AB, BC, CD, &c., of the polygon are equal chords; consequently they are equally distant from the center (Theo. VII. Bk. III.). Hence, if from the point *O*, as a center, and with the radius *O P*, a circle be described, the circumference will touch the side *B C*, and all the other sides of the polygon, each at its middle point, and the circle will be inscribed in the polygon (136).

225. *Scholium.* The point *O*, the common center of the circumscribed and inscribed circles, may also be regarded as the center of the polygon. The angle formed at the center by two radii drawn to the extremities of the same side is called *the angle at the center;* and the perpendicular from the center to a side is called the *apothegm* of the polygon.

226. *Cor.* 1. Since all the chords *A B*, *B C*, *C D*, &c. are equal, all the angles at the center must likewise be equal; therefore the value of each may be found by dividing four right angles by the number of sides of the polygon.



227. *Cor.* 2. If the inscribed polygon is a square, its side is to radius as the square root of 2 is to 1.

For, draw the diagonals *A C* and *B D*, and they will intersect at the center *O*, forming right angles, at that point. The triangle *A O B* is therefore right angled; and is also isosceles. Hence (Theo. IX. Cor. 3), we have

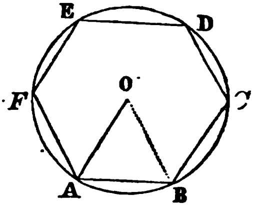
$$AB : AO :: \sqrt{2} : 1.$$

THEOREM XXIV.

228. *The side of a regular hexagon inscribed in a circle is equal to the radius of the circle.*

Let *A B C D E F* be a regular hexagon inscribed in a circle, the center of which is *O;* then any side, as *B C*, will be equal to the radius *O A*.

Join *B O;* and the angle at the center, *A O B* is one sixth of four right angles (Theo. XXIII. Cor, 1.), or one third of two

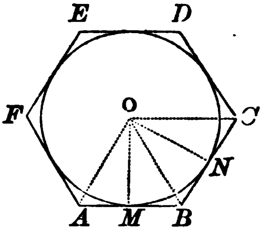


right angles; therefore the two other angles, *O A B*, *O B A*, of the same triangle, are together equal to two thirds of two right angles (Theo. XIX. Bk. I.). But *A O* and *B O* being equal, the angles *O A B*, *O B A* are also equal (Theo. VI. Bk. I.); consequently, each is one third of two right angles. Hence the triangle *A O B* is equiangular; therefore *A B*, the side of the regular hexagon, is equal to *A O*, the radius of the circle (Theo. VII. Cor. Bk. I.).

229. *Cor.* By joining the alternate angles of the inscribed regular hexagon by the straight lines *A C*, *C E*, *E A*, the figure *A C E*, thus inscribed in the circle, will be an equilateral triangle, since its sides subtend equal arcs, *A B C*, *C D E*, *E F A*, on the circumference (Theo. II. Bk. III.).

Theorem XXV.

230. *The area of a regular polygon is equivalent to the product of its perimeter by half of the radius of the inscribed circle.*



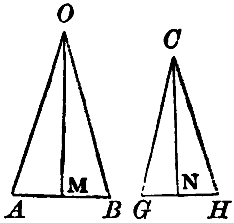
Let *A B C D E F* be a regular polygon, and *O* the center of the inscribed circle.

From *O* let the straight lines *O A*, *O B*, &c. be drawn to the vertices of all the angles of the polygon, and the polygon will be divided into as many equal triangles as it has sides; and let the radii *O M*, *O N*, &c. of the inscribed circle be drawn to the centers of the sides of the polygon, or to the points of tangency *M*, *N*, &c., and these radii are perpendicular to the sides respectively (Theo. VII. Cor. 1, Bk. III.); therefore the radius of the circle is equal to the altitude of the several triangles.

Now, the triangle *A O B* is measured by the product of *A B* by half of *O M* (Theo. VI.); the triangle *O B C* by the product of *B C* by the half of *O N*. But *O M* is equal to *O N;* hence the two triangles taken together are measured by the sum of *A B* and *B C* by half of *O M*. In like manner the measure of the other triangles may be found; hence, the sum of all the triangles, or the whole polygon, is equal to the sum of the bases *A B*, *B C*, &c., or the perimeter of the polygon, multiplied by half of *O M*, or half the radius of the inscribed circle.

THEOREM XXVI.

231. *The perimeters of two regular polygons, having the same number of sides, are to each other as the radii of the circumscribed circles, and, also, as the radii of the inscribed circles; and their areas are to each other as the squares of those radii.*



Let *A B* be a side of one polygon, *O* the center, and consequently *O A* the radius of the circumscribed circle, and *O M*, perpendicular to *A B*, the radius of the inscribed circle. Let *G H* be a side of the other polygon, *C* the center, *C G* and *C N* the radii of the circumscribed and the inscribed circles.

The perimeters of the two polygons are to each other as the sides *A B* and *G H* (Theo. XXI.), but the angles *A* and *G* are equal, being each half of the angle of the polygon; so also are the angles *B* and *H;* hence, drawing *O B* and *C H*, the isosceles triangles *A B O*, *G H C* are similar, as are likewise the right-angled triangles *A M O*, *G N C;* hence

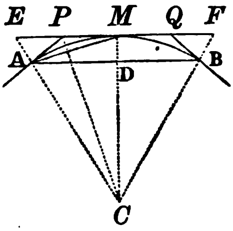
$$AB : GH :: AO . GC :: MO : NC.$$

Hence the perimeters of the polygons are to each other as the radii *A O*, *G C* of the circumscribed circles, and, also, as the radii *M O*, *N C* of the inscribed circles.

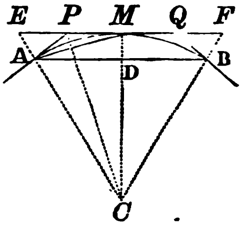
The surfaces of these polygons are to each other as the squares of the homologous sides AB, GH (Theo. XXI.); they are therefore to each other as the squares of AO, GC, the radii of the circumscribed circles, or as the squares of OM, CN, the radii of the inscribed circles.

THEOREM XXVII.

232. *The surface of a regular inscribed polygon, and that of a similar circumscribed polygon, being given; to find the surfaces of regular inscribed and circumscribed polygons having double the number of sides.*



Let AB be a side of the given inscribed polygon; EF, parallel to AB, a side of the circumscribed polygon, and C the center of the circle. Draw the chord AM, and the tangents AP, BQ; then AM will be a side of the inscribed polygon, having twice the number of sides; and PQ, the double of PM, will be a side of the similar circumscribed polygon.



Let A, then, be the surface of the inscribed polygon whose side is AB, B that of the similar circumscribed polygon; A' the surface of the polygon whose side is AM, B' that of the similar circumscribed polygon: A and B are given; we have to find A' and B':

First. The triangles ACD, ACM, whose common vertex is A, are to each other as their bases CD, CM (Theo. VI. Cor.); they are likewise as the polygons A and A'; hence

$$A : A' :: CD : CM.$$

Again the triangles CAM, CME, whose common vertex is M, are to each other as their bases CA, CE; they are likewise to each other as the polygons A' and B; hence

$$A' : B :: CA : CE.$$

But since AD and ME are parallel, we have,

$$CD : CM :: CA : CE;$$

hence

$$A : A' :: A' : B;$$

hence, *the polygon* A' *is a mean proportional between the two given polygons.*

Secondly. The altitude CM being common, the triangle CPM is to the triangle CPE as PM is to PE; but since CP bisects the angle MCE, we have (Theo. XII.),

$$PM : PE :: CM : CE :: CD : CA :: A : A';$$

hence

$$CPM : CPE :: A : A';$$

and, consequently,

$$CPM : CPM + CPE \text{ or } CME :: A : A + A'.$$

But $CMPA$ or $2\,CMP$ and CME are to each other as the polygons B' and B; hence

$$B' : B :: 2A : A + A';$$

which gives

$$B' = \frac{2A \times B}{A + A'};$$

or, *the polygon* B' *is equal to the quotient of twice the product of the given polygons divided by the sum of the inscribed polygons.*

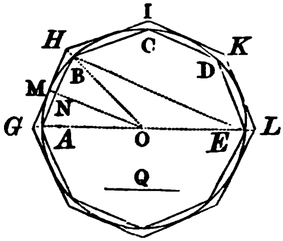
Thus, by means of the polygons A and B, it is easy to find the polygons A' and B', which have double the number of sides.

THEOREM XXVIII.

233. *A circle being given, two similar polygons can always be formed, the one circumscribed about the circle, the other inscribed in it, which shall differ from each other by less than any assignable surface.*

Let Q be the side of a square less than the given surface.

Bisect AC, a fourth part of the circumference, and then



bisect the half of this fourth, and so proceed until an arc is found whose chord *A B* is less than *Q*. As this arc must be an exact part of the circumference, if we apply the chords *A B*, *B C*, &c., each equal to *A B*, the last will terminate at *A*, and there will be inscribed in the circle a regular polygon, *A B C D E*, &c. Next describe about the circle a similar polygon, *G H I K L*, &c.; and the difference of these two polygons will be less than the square of *Q*.

Find the center, *O*; from the points *G* and *H* draw the straight lines *G O*, *H O*, through the points *A* and *B*. Draw also *O M* to the point of tangency, *M*; and it will bisect *A B* in *N*, and be perpendicular to it (Theo. V. Cor. 1, Bk. III.). Produce *A O* to *E*, and draw *B E*.

Let *P* represent the circumscribed polygon, and *p* the inscribed polygon. Then, since these polygons are similar, they are as the squares of the homologous sides *G H*, *A B* (Theo. XX.); but the triangles *G O H*, *A O B* are similar (Theo. XV.); hence they are to each other as the squares of the homologous sides *O G* and *O A* (Theo. XIX.); therefore

$$P : p :: \overline{OG}^2 : \overline{OA}^2 \text{ or } \overline{OM}^2.$$

Again, the triangles *O G M*, *E A B*, having their sides respectively parallel, are similar; therefore

$$P : p :: \overline{OG}^2 : \overline{OM}^2 :: \overline{AE}^2 : \overline{BE}^2;$$

and by division,

$$P : P - p :: \overline{AE}^2 : \overline{AE}^2 - \overline{EB}^2 \text{ or } \overline{AB}^2.$$

But *P* is less than the square described on the diameter *A E*; therefore *P* — *p* is less than the square described on *A B*, that is, less than the given square *Q*. Hence, the difference between the circumscribed and inscribed polygons may always be made less than any given surface.

234. *Cor.* Since the circle is obviously greater than any inscribed polygon, and less than any circumscribed one, it follows that *a polygon may be inscribed or circumscribed, which will differ from the circle by less than any assignable magnitude.*

Proposition XXIX.

235. *To find the approximate area of a circle whose radius is unity.*

Let the radius of the circle be 1, and let the first inscribed and circumscribed polygons be squares; the side of the inscribed square will be $\sqrt{2}$ (Theo. XXIII. Cor. 2), and that of the circumscribed square will be equal to the diameter 2. Hence the surface of the inscribed square is 2, and that of the circumscribed square is 4. Let, therefore, $A = 2$, and $B = 4$. Now it has been proved, in Theorem XXVII., that the surface of the inscribed octagon, or, as it has been represented, A, is a mean proportional between the two squares A and B, so that $A' = \sqrt{8} = 2.8284271$; and it has also been proved, in the same proposition, that the circumscribed octagon, represented by $B', = \frac{2A \times B}{A + A'}$; so that $B' = \frac{16}{2 + \sqrt{8}} = 3.3137085$. The inscribed and the circumscribed octagons being thus determined, we can easily, by means of them, determine the polygons having twice the number of sides. We have only in this case to put $A = 2.8284271$, $B = 3.3137085$; and we shall find $A' = \sqrt{A \times B} = 3.0614674$, and $B' = \frac{2A \times B}{A + A'} = 3.1825979$.

In like manner may be determined the area of polygons of sixteen sides, and thence the area of polygons of thirty-two sides, and so on till we arrive at an inscribed and a circumscribed polygon differing so little from each other, and consequently from the circle, that the difference shall be less than any assignable magnitude (Theo. XXVII. Cor.).

The numerical expression for the surface, of these polygons of 32768 sides, agreeing as far as the seventh order of decimals, is 3.1415926.

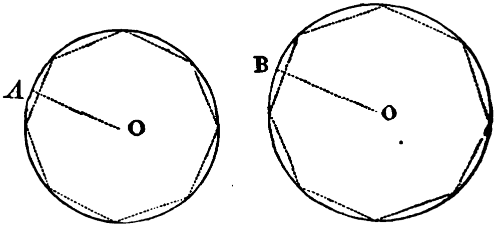
As the circle is between the inscribed and circumscribed polygon of 32768 sides, it cannot, strictly speaking, differ from either so much as they do from each other; so that the number 3.1415926 expresses the *area* of a circle whose radius is 1 correctly, as far as seven orders of decimals.

236. *Cor.* Since the inscribed and circumscribed polygons are regular, and have the same number of sides, they are similar (Theo. I); therefore, by increasing the number of the sides, the corresponding polygons formed will approach to an equality with the circle. Now if, by continual bisections, the polygons formed shall have their number of sides indefinitely great, each side will become indefinitely small, and the inscribed and circumscribed polygons will ultimately coincide with each other. But when they coincide with each other, they must each coincide with the circle, since no part of an inscribed polygon can be without the circle, nor can any part of a circumscribed one be within it; hence, *the perimeters of the polygons must coincide with the circumference of the circle, and be equal to it.*

237. *Scholium.* Every circle, therefore, may be regarded as a polygon of an infinite number of sides.

THEOREM XXX

238. *The circumferences of circles are to each other as their radii, and their areas are to each other as the squares of their radii.*



Let C denote the circumference of one of the circles, R its radius OA, A its area; and let C' denote the circumference of the other circle, r its radius OB, A' its area; then will

$$C : C' :: R : r,$$

and $$A : A' :: R^2 : r^2.$$

Inscribe within the given circles two regular polygons of the same number of sides; and, whatever be the number of sides, the perimeters of the polygons will be to each other as the radii OA and OB (Theo. XXVI.). Now, conceive the arcs subtending the sides of the polygon to be continually bisected, forming other inscribed polygons, until polygons are formed of an indefinite number of sides, and therefore having perimeters coinciding with the circumference of the circumscribed circles (Prop. XXIX. Cor.); and we shall have

$$C : C' :: R : r.$$

Again, the areas of the inscribed polygons are to each other as $\overline{OA}^2$ to $\overline{OB}^2$ (Theo. XXVI.). But when the number of sides of the polygons is indefinitely increased, the areas of the polygons become equal to the areas of the circles; hence we shall have

$$A : A' :: R^2 : r^2.$$

239. *Cor.* 1. The circumferences of circles are to each other as twice their radii, or as their diameters.

For, multiplying the terms of the second ratio in the first proportion by 2, we have

$$C : C' :: 2R : 2r.$$

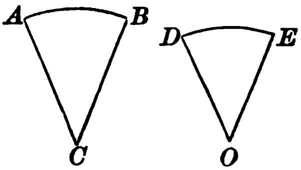
240. *Cor.* 2. The areas of circles are to eacn other as the squares of their diameters.

For, multiplying the second ratio of the second proportion by 4, or 2 squared, we have

$$A : A' :: 4R^2 : 4r^2.$$

THEOREM XXXI.

241. *Similar arcs are to each other as their radii; and similar sectors are to each other as the squares of their radii.*



Let $A\,B$, $D\,E$ be similar arcs; $A\,C\,B$, $D\,O\,E$, similar sectors; and denote the radii $C\,A$ and $O\,D$ by R and r; then will

$$A\,B : D\,E :: R : r,$$

and

$$A\,C\,B : D\,O\,E :: R^2 : r^2.$$

For, since the arcs are similar, the angle C is equal to the angle O (175). But the angle C is to four right angles as the arc $A\,B$ is to the whole circumference described with the radius $C\,A$ (Theo. IX. Sch. 2, Bk. III.); and the angle O is to four right angles as the arc $D\,E$ is to the circumference described with the radius $O\,D$. Hence, the arcs $A\,B$, $D\,E$ are to each other as the circumferences of which they form a part. But these circumferences are to each other as their radii, $C\,A$, $O\,D$ (Theo. XXX.); therefore

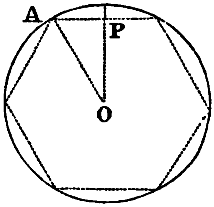
$$\textit{Arc}\ A\,B : \textit{Arc}\ D\,E :: R : r.$$

By like reasoning, the sectors $A\,C\,B$, $D\,O\,E$ are to each other as the whole circles of which they are a part; and these are the squares of their radii (Theo. XXX.); therefore

$$\textit{Sector}\ A\,C\,B : \textit{Sector}\ D\,O\,E :: R^2 : r^2.$$

THEOREM XXXII.

242. *The area of a circle is equal to the product of the circumference by half the radius.*

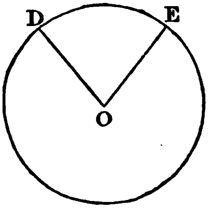


Let C denote the circumference of the circle, whose center is O, R its radius $O\,A$, and A its area; then will

$$A = C \times \tfrac{1}{2} R.$$

For, inscribe in the circle any regular polygon, and from the center draw $O\,P$ perpendicular to one of the sides. The area of the polygon, whatever the number of sides, will be equal to its perimeter multiplied by half of $O\,P$ (Theo. XXV.). Conceive the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will be equal to the circumference of the circle (Theo. XXIX. Cor.), and OP be equal to the radius OA; therefore the area of the polygon is equal to that of the circle; hence

$$A = C \times \tfrac{1}{2} R.$$



243. *Cor.* 1. The area of a sector is equal to the product of its arc by half of its radius.

For, let C denote the circumference of the circle of which the sector $D\,O\,E$ is a part, R its radius OD, and A its area; then we shall have (Theo. IX. Sch. 2, Bk. III.),

$$\textit{Sector } D\,O\,E : A :: \textit{Arc } D\,E : C;$$

hence, since equimultiples of two magnitudes have the same ratio as the magnitudes themselves (Theo. VIII. Bk. II.),

$$\textit{Sector } D\,O\,E : A :: \textit{Arc } D\,E \times \tfrac{1}{2} R : C \times \tfrac{1}{2} R.$$

But A, or the area of the whole circle, is equal to $C \times \tfrac{1}{2} R$; hence the area of the sector $D\,O\,E$ is equal to the arc $D\,E \times \tfrac{1}{2} R$.

244. *Cor.* 2. Let the circumference of the circle whose diameter is unity be denoted by π (which is called *pi*), the radius by R, and the diameter by D; and the circumference of any other circle by C, and its area by A. Then, since circumferences are to each other as their diameters (Theo. XXX. Cor. 1), we shall have,

$$C : D :: \pi : 1;$$

therefore

$$C = D \times \pi = 2\,R \times \pi.$$

Multiplying both numbers of this equation by $\tfrac{1}{2} R$, we have

$$C \times \tfrac{1}{2} R = R^2 \times \pi, \text{ or } A = R^2 \times \pi;$$

that is, *the area of a circle is equal to the product of the square of its radius by the constant number* π.

245. *Cor.* 3. The circumference of every circle is equal to the product of its diameter, or twice its radius, by the constant number π.

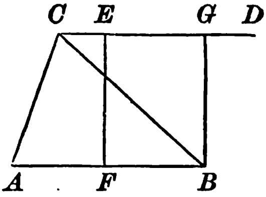
246. *Cor.* 4. The constant number π denotes the ratio of the circumference of any circle to its diameter; for $\frac{C}{D} = \pi$.

247. *Scholium.* The exact numerical value of the ratio denoted by π can be only approximately expressed. The approximate value found by Proposition XXIX. is 3.1415926; but, for most practical purposes, it is sufficiently accurate to take $\pi = 3.1416$. The symbol π is the first letter of the Greek word *περίμετρον*, *perimetron*, which signifies *circumference.*

PROBLEMS IN CONSTRUCTION.

248. The following Problems depend, upon the application of principles established in what has preceded, for solution: —

PROBLEM I. *To construct a rectangle that shall be equivalent to a given triangle.*

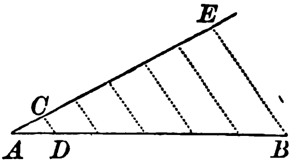


Let $A B C$ be the given triangle. Draw the indefinite straight line $C D$ parallel to the base $A B$; bisect $A B$ by the perpendicular $E F$, and make $E G$ equal to $F B$. Then, by drawing $G B$, the rectangle $E F B G$ is equal to the triangle $A B C$.

For the rectangle $E F B G$ has the same altitude, $E F$, as the triangle $A B C$, and half its base (Theo. II. Cor. 1, Bk. IV.).

PROBLEM II. *To divide a given straight line into any number of equal parts.*

Let $A B$ be the given straight line proposed to be divided into any number of equal parts; for example, six.



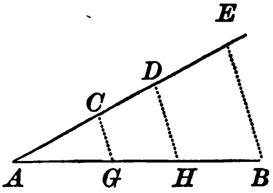
Through the extremity A draw the indefinite straight line $A\,E$, making any angle with $A\,B$. Take $A\,C$ of any convenient length, and apply it six times upon $A\,E$. Join the last point of division, E, and the extremity B by the straight line $E\,B$; and through the point C draw $C\,D$ parallel to $E\,B$; then $A\,D$ will be the sixth part of the line $A\,B$, and, being applied six times to $A\,B$, divides it into six equal parts.

For, since $C\,D$ is parallel to $E\,B$, in the triangle $A\,B\,E$, we have the proportion (Theo. X. Bk. IV.),

$$A\,D : A\,B :: A\,C : A\,E.$$

But $A\,C$ is the sixth part of $A\,E$; hence $A\,D$ is the sixth part of $A\,B$.

Problem III. *To divide a given straight line into parts that shall be proportional to other given lines.*



Let $A\,B$ be the given straight line proposed to be divided into parts proportional to the given lines $A\,C$, $C\,D$, $D\,E$.

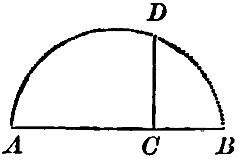
Through the point A draw the indefinite straight line $A\,E$, making any angle with $A\,B$. On $A\,E$ lay off $A\,C$, $C\,D$, and $D\,E$. Join the points E and B by the straight line $E\,B$, and through the points C and D draw $C\,G$ and $D\,H$ parallel to $E\,B$; and the line $A\,B$ will be divided into parts proportional to the given lines.

For, since $C\,G$ and $D\,H$ are each parallel to $E\,B$, we have the proportion (Theo. X. Cor. 2, Bk. IV.),

$$A\,C : A\,G :: C\,D : G\,H : D\,E : H\,B.$$

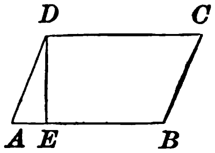
Problem IV. *To find a mean proportional between two given straight lines.*

Draw the indefinite straight line $A\,B$. On $A\,B$ take $A\,C$



equal to the first of the given lines, and CB equal to the second. On AB, as a diameter, describe a semicircle, and at the point C draw the perpendicular CD, meeting the semi-circumference in D; then, in the triangle ADB, will CD be a mean proportional to AC and CB (Theo. XVII. Cor. 2, Bk. IV.).

Problem V. *To construct a square that shall be equivalent to a given parallelogram, or to a given triangle.*



First. Let $ABCD$ be the given parallelogram, AB its base, and DE its altitude.

Find a mean proportional between AB and DE (Prob. IV.); and the square constructed on that proportional will be equivalent to the parallelogram $ABCD$.

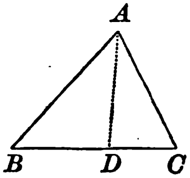
For, denoting the mean proportional by xy, we have, by construction,

$$AB : xy :: xy : DE;$$

therefore,

$$\overline{xy}^2 = AB \times DE;$$

but $AB \times DE$ is the measure of the parallelogram, and $\overline{xy}^2$ that of the square; hence they are equivalent.



Secondly. Let ABC be the given triangle, BC its base, and AD its altitude.

Find a mean proportional between BC and the half of AD, and let xy denote that proportional; the square constructed on xy will be equivalent to the triangle ABC.

For since, by construction,

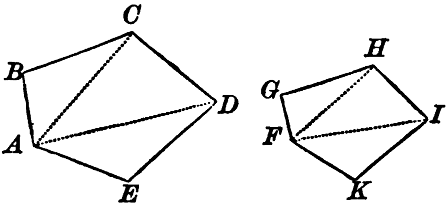
$$BC : xy :: xy : \tfrac{1}{2} AD,$$

it follows that

$$\overline{xy}^2 = BC \times \tfrac{1}{2} AD;$$

hence the square constructed on xy is equivalent to the triangle ABC.

Problem VI. *Upon a given straight line to construct a polygon similar to a given polygon.*



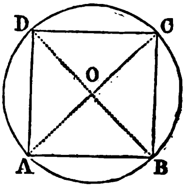
Let $ABCDE$ be the given polygon, and FG the given straight line.

Draw the diagonals AC, AD. At the point F in the straight line FG make the angle GFH equal to the angle BAC; and at the point G make the angle FGH equal to the angle ABC. The lines FH, GH will cut each other in H, and FGH will be a triangle similar to ABC. In the same manner, upon FH, homologous to AC, construct the triangle FIH similar to ADC; and upon FI, homologous to AD, construct the triangle FIK similar to ADE. The polygon $FGHIK$ will be similar to $ABCDE$, as required.

For these two polygons are composed of the same number of triangles, similar each to each, and similarly situated (Theo. XX. Cor., Bk. IV.).

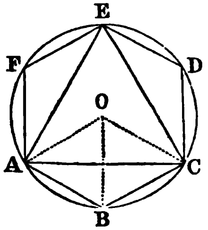
Problem VII. *To inscribe a square in a given circle.*

Draw two diameters, AC, BD, intersecting each other at right angles; join their extremities, A, B, C, D, and the figure $ABCD$ will be a square.



For, the angles AOB, BOC, &c. being equal, the chords AB, BC, &c. are also equal (Theo. II. Bk. III.); and the angles ABC, BCD, &c., being inscribed in semicircles, are right angles (Theo. X. Cor. 2, Bk. III.). Hence $ABCD$ is a square, and it is inscribed in the circle $ABCD$.

Problem VIII. *To inscribe a hexagon in a given circle.*



Begin with point A, and apply the radius, $A\,O$, of the circle six times as a chord to the circumference, we are brought round to the point of beginning, and the inscribed figure $A\,B\,C\,D\,E\,F$, thus formed, is a regular hexagon.

Cor. Join the alternate angles of the inscribed regular hexagon by the straight lines $A\,C$, $C\,E$, $E\,A$, the figure $A\,C\,E$, thus inscribed in the circle, will be an equilateral triangle, since its sides subtend equal arcs, $A\,B\,C$, $C\,D\,E$, $E\,F\,A$, on the circumference (Theo. II. Bk. III.).

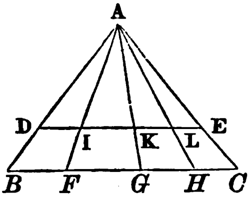
EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

249. 1. Equivalent triangles upon the same base, and upon the same side of a straight line, are between the same parallels.

2. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equivalent to, the given triangle.

3. If the sides of the square described upon the hypothenuse of a right-angled triangle be produced to meet the sides (produced if necessary) of the squares described upon the other two sides of the triangle, the triangles thus formed will be similar to the given triangle, and two of them will be equal to it.



4. In any triangle, if a line be drawn parallel to the base, all the lines drawn from the vertex will divide the parallel and the base proportionally.

5. A square circumscribed about a given circle is double a square inscribed in the same circle.

6. Straight lines drawn from the vertices of a triangle, so as to bisect the opposite sides, bisect also the triangle.

7. The side of a square circumscribed about a circle is equal to the diagonal of a square inscribed in the same circle.

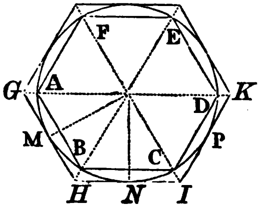
8. To describe a circle in a square.

9. To construct a rectangle that shall be equivalent to a given triangle.

10. To find the side of a square that shall be equivalent to the sum of two given squares.

11. A regular inscribed polygon being given, to circumscribe a similar polygon about the same circle.

12. A regular circumscribed polygon being given, to inscribe a similar polygon within the same circle.



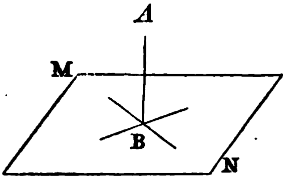
BOOK V.*

PLANES AND POLYEDRONS.

DEFINITIONS.

250. A straight line is *perpendicular to a plane*, when it is perpendicular to every straight line which it meets in that plane.

Conversely, the plane, in the same case, is *perpendicular to the line.*



The **Foot** of the perpendicular is the point in which it meets the plane.

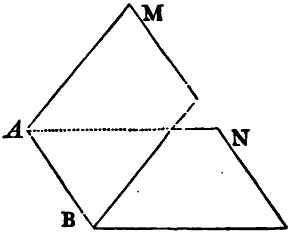
Thus the straight line AB is perpendicular to the plane MN; the plane MN is perpendicular to the straight line AB; and B is the foot of the perpendicular AB.

251. A line is *parallel to a plane*, when it cannot meet the plane, however far both of them may be produced.

Conversely, the plane, in the same case, is parallel to the line.

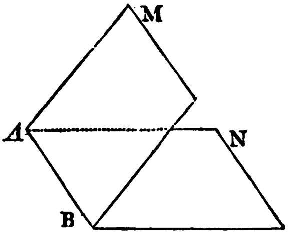
252. Two *planes are parallel to each other*, when they cannot meet, however far both of them may be produced.

253. A **Diedral Angle** is an angle formed by the intersection of two planes, and is measured by the inclination of two straight lines drawn from any point in the line of intersection, perpendicular to that line, one being drawn in each plane.



---

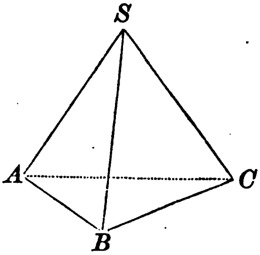
* Books V. and VI. can be omitted.



The line of common section is called the *edge*, and the two planes are called the faces, of the diedral angle.

Thus the two planes $A\,B\,M$, $A\,B\,N$, whose line of intersection is $A\,B$, form a diedral angle, of which the line $A\,B$ is the edge, and the planes $A\,B\,M$, $A\,B\,N$ are the faces.

254. A diedral angle may be acute, right, or obtuse. If the two faces are perpendicular to each other, the angle is right.



255. A Polyedral Angle is an angle formed by the meeting at one point of more than two plane angles, which are not in the same plane.

The common point of meeting of the planes is called the *vertex*, each of the plane angles a *face*, and the line of common section of any two of the planes an *edge* of the polyedral angle.

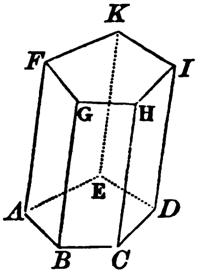
Thus the three plane angles $A\,S\,B$, $B\,S\,C$, $C\,S\,A$ form a polyedral angle, whose vertex is S, whose faces are the plane angles, and whose edges are the sides, $A\,S$, $B\,S$, $C\,S$, of the same angles.

256. A polyedral angle formed by three faces is called a *triedral* angle; by four faces, a *tetraedral;* by five faces, a *pentaedral*, &c.

257. A Polyedron is a solid, or volume, bounded by planes.

The bounding planes are called the *faces* of the polyedron; and the lines of intersection of the faces are called the *edges* of the polyedron.

258. A Prism is a polyedron having two of its faces equal and parallel polygons, and the other faces parallelograms.



The equal and parallel polygons are called the *bases* of the prism, and the parallelograms its *lateral* faces. The lateral faces taken together constitute the *lateral* or *convex surface* of the prism.

Thus the polyedron $A\,B\,C\,D\,E-K$ is a prism, having for its bases the equal and parallel polygons $A\,B\,C\,D\,E$, $F\,G\,H\,I\,K$, and for its lateral faces the parallelograms $A\,B\,G\,F$, $B\,C\,H\,G$, &c.

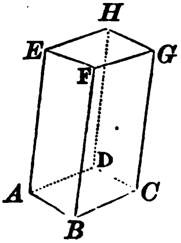
The *principal edges* of a prism are those which join the corresponding angles of the bases; as $A\,F$, $B\,G$, &c.

259. The **Altitude** of a prism is a perpendicular drawn from any point in one base to the plane of the other.

260. A **Right Prism** is one whose principal edges are perpendicular to the planes of its bases. Each of the edges is then equal to the altitude of the prism. Every other prism is *oblique*, and has each edge greater than the altitude.

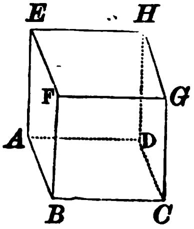
261. A prism is *triangular*, *quadrangular*, *pentangular*, *hexangular*, &c., according as its base is a triangle, a quadrilateral, a pentagon, a hexagon, &c.

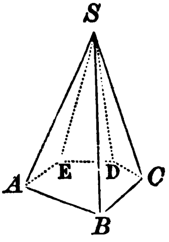
262. A **Parallelopipedon** is a prism whose bases are parallelograms; as the prism $A\,B\,C\,D-H$.



The parallelopipedon is *rectangular* when all its faces are rectangles; as the parallelopipedon $A\,B\,C\,D-H$.

263. A **Cube,** or **Regular Hexaedron,** is a rectangular parallelopipedon having all its faces equal squares; as the parallelopipedon $A\,B\,C\,D-H$.





264. A **Pyramid** is a polyedron of which one of the faces is any polygon, and all the others are triangles meeting at a common point.

The polygon is called the *base* of the pyramid, the triangles its *lateral faces*, and the point at which the triangles meet its *vertex*. The lateral faces taken together constitute the *lateral* or *convex surface* of the pyramid.

Thus the polyedron $ABCDE-S$ is a pyramid, having for its base the polygon $ABCDE$, for its lateral faces the triangles ASB, BSC, CSD, &c., and for its vertex the point S.

265. The **Altitude** of a pyramid is a perpendicular drawn from the vertex to the plane of the base.

266. A pyramid is *triangular*, *quadrangular*, &c., according as its base is a triangle, a quadrilateral, &c.

267. A **Right Pyramid** is one whose base is a regular polygon, and the perpendicular drawn from the vertex to the base passes through the center of the base. In this case the perpendicular is called the *axis* of the pyramid.

268. The **Slant Height** of a right pyramid is a line drawn from the vertex to the middle of one of the sides of the base.

269. A **Frustum** of a pyramid is the part of the pyramid included between the base and a plane cutting the pyramid parallel to the base.

270. The **Altitude** of the frustum of a pyramid is the perpendicular distance between its parallel bases.

271. The **Slant Height** of a frustum of a right pyramid is that part of the slant height of the pyramid which is intercepted between the bases of the frustum.

272. The **Axis** of the frustum of a pyramid is that of the axis of the pyramid which is intercepted between the bases of the frustum.

273. The **Diagonal** of a polyedron is a line joining the vertices of any two of its angles which are not in the same face.

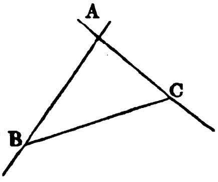
274. **Similar Polyedrons** are those which are bounded by the same number of similar faces, and have their polyedral angles respectively equal.

275. A **Regular Polyedron** is one whose faces are all equal and regular polygons, and whose polyedral angles are all equal to each other.

276. The **Volume** of a polyedron is its measured magnitude, or the number of times it contains some other polyedron as a unit.

THEOREM I.

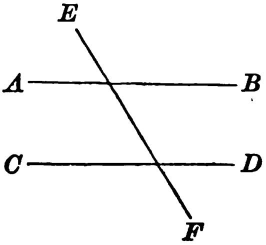
277. *Two straight lines which intersect each other lie in the same plane and determine its position.*



Let $A\,B$, $A\,C$ be two straight lines which intersect each other in A; then these lines will be in the same plane.

Conceive a plane to pass through $A\,B$, and to be turned about $A\,B$, until it pass through the point C; then, the two points A and C being in this plane, the line $A\,C$ lies wholly in it (Art. 7). Hence, the position of the plane is determined by the condition of its containing the two straight lines $A\,B$, $A\,C$.

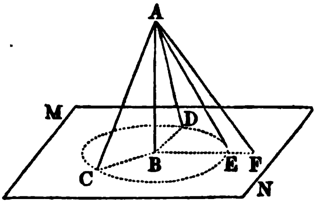
278. *Cor.* 1. A triangle, $A\,B\,C$, or three points, A, B, C, not in a straight line, determine the position of a plane.



279. *Cor.* 2. Hence, also, two parallels, $A\,B$, $C\,D$, determine the position of the plane; for, drawing the secant $E\,F$, the plane of the two straight lines $A\,B$, $E\,F$ is that of the parallels $A\,B$, $C\,D$.

THEOREM II.

280. *Oblique lines drawn from a point to a plane at equal distances from a perpendicular drawn from the same point to it, are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the longer.*



Let $A\,B$ be perpendicular to the plane $M\,N$; and $A\,C$, $A\,D$, $A\,E$ be oblique lines, from the point A, meeting the plane at equal distances, $B\,C$, $B\,D$, $B\,E$, from the perpendicular; and $A\,F$ a line meeting the plane more remote from the perpendicular; then will $A\,C$, $A\,D$, $A\,E$ be equal to each other, and $A\,F$ be longer than $A\,C$.

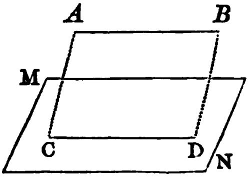
For, the angles $A\,B\,C$, $A\,B\,D$, $A\,B\,E$ being right angles, and the distances $B\,C$, $B\,D$, $B\,E$ being equal to each other, the triangles $A\,B\,C$, $A\,B\,D$, $A\,B\,E$ have in each an equal angle contained by equal sides; consequently they are equal (Theo. IV. Bk. I.); therefore, the hypothenuses, or the oblique lines $A\,C$, $A\,D$, $A\,E$, are equal to each other.

In like manner, since the distance $B\,F$ is greater than $B\,C$, or its equal $B\,E$, the oblique line $A\,F$ must be greater than $A\,E$, or its equal $A\,C$ (Theo. X. Bk. I.).

281. *Cor.* All the equal oblique lines $A\,C$, $A\,D$, $A\,E$, &c. terminate in the circumference of a circle, $C\,D\,E$, described from B, the foot of the perpendicular, as a center; therefore, a point, A, being given out of a plane, the point B, at which the perpendicular let fall from it would meet that plane, may be found by taking upon the plane three points, C, D, E, equally distant from the point A, and then finding the center of the circle which passes through these points; this center will be the point B required.

THEOREM III.

282. *If a straight line without a plane is parallel to a line within the plane, it is parallel to the plane.*

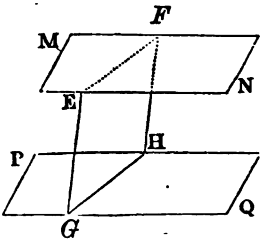


Let the straight line *A B*, without the plane *M N*, be parallel to the line *C D* in that plane; then will *A B* be parallel to the plane *M N*.

Conceive a plane *A B C D* to pass through the parallels *A B*, *C D*. Now, if the line *A B*, which lies in the plane *A B C D*, could meet the plane *M N*, it could only be in some point of the line *C D*, the common section of the two planes; but the line *A B* cannot meet *C D*, since they are parallel (14); therefore it will not meet the plane *M N*; hence it is parallel to that plane (251).

THEOREM IV.

283. *If two parallel planes are cut by a third plane, the two intersections are parallel.*

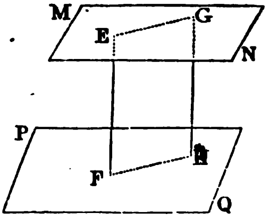


Let the two parallel planes *M N* and *P Q* be cut by the plane *E F G H*, and let their intersections with it be *E F*, *G H*; then *E F* is parallel to *G H*.

For, if the lines *E F*, *G H*, lying in the same plane were not parallel, they would meet each other on being produced; therefore the planes *M N*, *P Q*, in which those lines are situated, would also meet, which is impossible, since these planes are parallel.

THEOREM V.

284. *Parallel straignt lines included between two parallel planes are equal.*

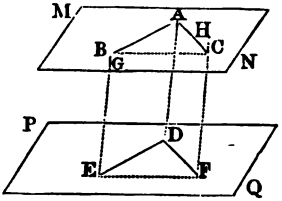


Let *E F*, *G H* be two parallel straight lines, included between two parallel planes, *M N*, *P Q;* then *E F* and *G H* are equal.

For, through the parallels *E F*, *G H* conceive the plane *E F G H* to pass, intersecting the parallel planes in *E G*, *F H*. The intersections *E G*, *F H* are parallel to each other (Theorem IV.); and *E F*, *G H* are also parallel; consequently the figure *E F G H* is a parallelogram; hence *E F* is equal to *G H* (Theo. XXII. Bk. I.).

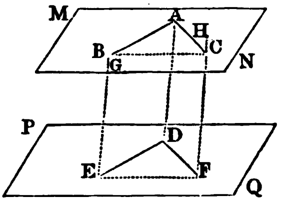
Theorem VI.

285. *If two angles not in the same plane have their sides parallel and lying in the same direction, these angles will be equal, and their planes will be parallel.*



Let *B A C*, *E D F* be two triangles, lying in different planes, *M N* and *P Q*, having their sides parallel and lying in the same direction; then the angles *B A C*, *E D F* will be equal, and their planes, *M N*, *P Q*, be parallel.

For, take *A B* equal to *E D*, and *A C* equal to *D F;* and join *B C*, *E F*, *B E*, *A D*, *C F*. Since *A B* is equal and parallel to *E D*, the figure *A B E D* is a parallelogram (Theo. XXIV. Bk. I.); therefore *A D* is equal and parallel to *B E*. For a similar reason, *C F* is equal and parallel to *A D;* hence, also, *B E* is equal and parallel to *C F;* hence the figure *B C F E* is a parallelogram, and the side *B C* is equal and parallel to *E F;* therefore the triangles *B A C*, *E D F* have their sides equal, each to each; hence the angle *B A C* is equal to the angle *E D F*.

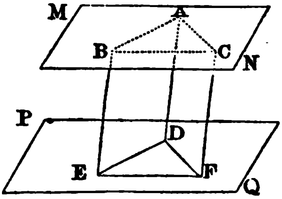


Again, the plane *B A C* is parallel to the plane *E D F*. For, if not, suppose a plane to pass through the point *A*, parallel to *E D F*, meeting the lines *B E*, *C F*, in points different from *B* and *C*, for instance *G* and *H*. Then the three lines *G E*, *A D*, *H F* will be equal (Theo. V.). But the three lines *B E*, *A D*, *C F* are already known to be equal; hence *B E* is equal to *G E*, and *H F* is equal to *C F*, which is absurd; hence the plane *B A C* is parallel to the plane *E D F*.

286. *Cor.* If two parallel planes, *M N*, *P Q*, are met by two other planes, *A B E D*, *A C F D*, the angles *B A C*, *E D F*, formed by the intersections of the parallel planes, are equal; for the intersection *A B* is parallel to *E D*, and *A C* to *D F* (Theo. IV.); therefore the angle *B A C* is equal to the angle *E D F*.

THEOREM VII.

287. *If three straight lines not in the same plane are equal and parallel, the triangles formed by joining the extremities of these lines will be equal, and their planes will be parallel.*

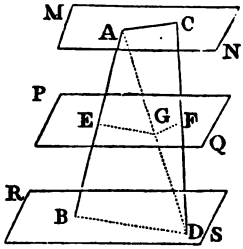


Let *B E*, *A D*, *C F* be three equal and parallel straight lines, not in the same plane, and let *B A C*, *E D F* be two triangles formed by joining the extremities of these lines; then will these triangles be equal, and their planes parallel.

For, since *B E* is equal and parallel to *A D*, the figure *A B E D* is a parallelogram; hence, the side *A B* is equal and parallel to *D E* (Theo. XXIV. Bk. I.). For a like reason, the sides *B C*, *E F* are equal and parallel; so also are *A C*, *D F*; hence, the two triangles *B A C*, *E D F*, having their sides equal, are themselves equal (Theo. XIII. Bk. I.); consequently, as shown in the last proposition, their planes are parallel.

THEOREM VIII.

288. *If two straight lines are cut by three parallel planes, they will be divided proportionally.*



Let the straight line AB meet the parallel planes MN, PQ, RS, at the points A, E, B; and the straight line CD meet the same planes at the points C, F, D; then will

$$AE : EB :: CF : FD.$$

Draw the line AD, meeting the plane PQ in G, and draw AC, EG, BD. Then the two parallel planes PQ, RS, being cut by the plane ABD, the intersections EG, BD are parallel (Theo. IV.); and, in the triangle ABD, we have (Theo. X. Bk. IV.),

$$AE : EB :: AG : GD.$$

In like manner, the intersections AC, GF being parallel, in the triangle ADC, we have

$$AG : GD :: CF : FD;$$

hence, since the ratio $AG : GD$ is common to both proportions, we have

$$AE : EB :: CF : FD.$$

THEOREM IX.

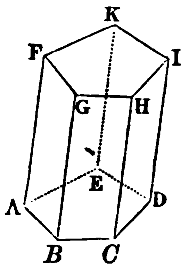
289. *The convex surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*

Let $ABCDE-K$ be a right prism; then will its convex surface be equal to the perimeter of its base,

$$AB + BC + CD + DE + EA$$

multiplied by its altitude AF.

For, the convex surface of the prism is equal to the sum of the parallelograms AG, BH, CI, DK, EF (258). Now,



the area of each of those parallelograms is equal to its base, AB, BC, CD, &c., multiplied by its altitude, AF, BG, CH, &c. (Theo. V. Bk. IV.). But the altitudes AF, BG, CH, &c. are each equal to AF, the altitude of the prism. Hence, the area of these parallelograms, or the convex surface of the prism, is equal to

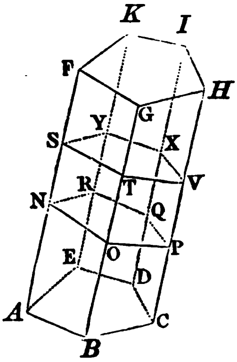
$$(AB+BC+CD+DE+EA)\times AF;$$

or the product of the perimeter of the prism by its altitude.

290. *Cor.* If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

THEOREM X.

291. *In every prism, the sections formed by parallel planes are equal polygons.*



Let the prism $ABCDE-K$ be intersected by the parallel planes NP, SV; then are the sections $NOPQR$, $STVXY$ equal polygons.

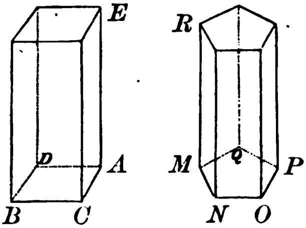
For the sides ST, NO are parallel, being the intersections of two parallel planes with a third plane $ABGF$ (Theo. IV.); these same sides ST, NO, are included between the parallels NS, OT, which are sides of the prism; hence NO is equal to ST. For like reasons, the sides OP, PQ, QR, &c. of the section $NOPQR$, are respectively equal to the sides TV, VX, XY, &c. of the section $STVXY$; and since the equal sides are at the same time parallel, it follows that the angles NOP, OPQ, &c. of the first section are respectively equal to the angles STV, TVX of the second

(Theo. VI.). Hence, the two sections $NOPQR$, $STVXY$, are equal polygons.

292. *Cor.* Every section made in a prism parallel to its base, is equal to that base.

THEOREM XI.

293. *Prisms having equivalent bases and equal altitudes are equivalent.*



Let $E-ABCD$ and $R-MNOPQ$ be two prisms having equivalent bases and equal altitudes; then they are equivalent.

For, any sections made by planes respectively parallel to the bases $ABCD$ and $MNOPQ$, are equivalent to those bases (Theo. X.); and these latter are equivalent to each other by hypothesis. Now the prisms may be regarded as being made up of sections parallel to their bases, and of the same number in each unit of height; then, since the altitudes of the prisms are equal, the number of tiers of the sections in each prism must be equal; therefore, the prisms must be equivalent.

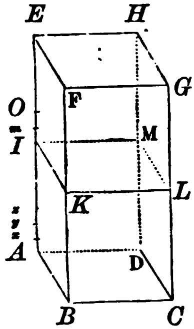
294. *Cor.* Any prism is equivalent to a right prism having an equivalent base and equal altitude.

THEOREM XII.

295. *Two rectangular parallelopipedons, which have the same base, are to each other as their altitudes.*

Let the two parallelopipedons AG, AL have the same base, $ABCD$; then they are to each other as their altitudes, AE, AI.

First. Suppose the altitudes AE, AI are to each other as two whole numbers; for example, as 15 is to 8. Divide AE



into 15 equal parts, of which AI will contain 8. Through x, y, z, &c., the points of division, conceive planes to pass parallel to the common base. These planes will divide the solid AG into 15 small parallelopipedons, all equal to each other, having equal bases and equal altitudes; equal bases, since every section, as $IKLM$, parallel to the base $ABCD$, is equal to that base (Theo. X.), and equal altitudes, since the altitudes are the equal divisions Ax, xy, yz, &c. But of those 15 equal parallelopipedons, 8 are contained in AL; hence the parallelopipedon AG is to the parallelopipedon AL as 15 is to 8, or, in general, as the altitude AE is to the altitude AI.

Secondly. If the ratio of AE to AI cannot be exactly expressed by numbers, we shall still have the proportion,

$$\textit{Solid } AG : \textit{Solid } AL :: AE : AI.$$

For, if this proportion is not correct, suppose we have

$$\textit{Solid } AG : \textit{Solid } AL :: AE : AO \text{ greater than } AI.$$

Divide AE into equal parts, each of which shall be less than IO; there will be at least one point of division, m, between I and O. Let P represent the parallelopipedon, whose base is $ABCD$, and altitude Am; since the altitudes AE, Am are to each other as two whole numbers, we shall have

$$\textit{Solid } AG : P :: AE : Am.$$

But, by hypothesis, we have

$$\textit{Solid } AG : \textit{Solid } AL :: AE : AO;$$

hence (Theo. IX. Cor. 2, Bk. II.),

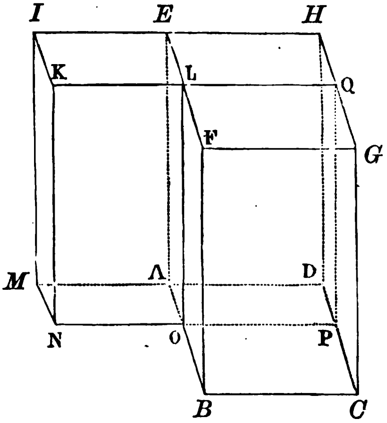
$$\textit{Solid } AL : P :: AO : Am.$$

But AO is greater than Am; hence, if the proportion is correct, the parallelopipedon AL must be greater than P. On the contrary, however, it is less; consequently the solid AG cannot be to the solid AL as the line AE is to a line greater than AI.

By the same mode of reasoning, it may be shown that the fourth term of the proportion cannot be less than AI; therefore it must be equal to AI. Hence rectangular parallelopipedons, having the same base, are to each other as their altitudes.

THEOREM XIII.

296. *Two rectangular parallelopipedons, having the same altitude, are to each other as their bases.*



Let the two rectangular parallelopipedons AG, AK have the same altitude, AE; then they are to each other as their bases.

Place the two solids so that their faces, BE, OE, may have the common angle BAE; produce the plane $ONKL$ till it meets the plane $DCGH$ in PQ; we shall thus have a third parallelopipedon, AQ, which may be compared with each of the parallelopipedons AG, AK. The two solids, AG, AQ, having the same base, $AEHD$, are to each other as their altitudes AB, AO (Theo. XII.); in like manner, the two solids AQ, AK, having the same base, $AOLE$, are to each other as their altitudes AD, AM. Hence we have the two proportions,

$$\text{Solid } AG : \text{Solid } AQ :: AB : AO,$$
$$\text{Solid } AQ : \text{Solid } AK :: AD : AM.$$

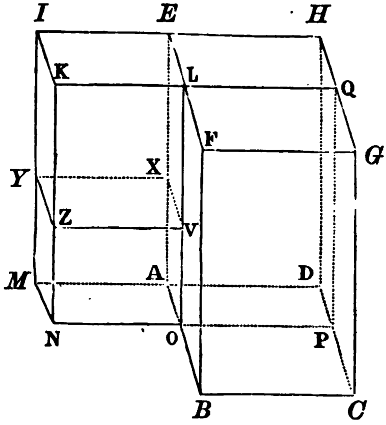
Multiplying together the corresponding terms of these proportions, and omitting, in the result, the common factor *Solid* AQ, we shall have,

$$\text{Solid } AG : \text{Solid } AK :: AB \times AD : AO \times AM.$$

But $AB \times AD$ measures the base $ABCD$ (Theo. IV. Sch., Bk. IV.); and $AO \times AM$ measures the base $AMNO$; hence two rectangular parallelopipedons of the same altitude are to each other as their bases.

THEOREM XIV.

297. *Any two rectangular parallelopipedons are to each other as the product of their bases by their altitudes.*



Let AG, AZ be two rectangular parallelopipedons; then they are to each other as the product of their bases, $ABCD$, $AMNO$, by their altitudes, AE, AX.

Place the two solids so that their faces, BE, OX, may have the common angle BAE; produce the planes necessary for completing the third parallelopipedon, AK, having the same altitude with the parallelopipedon AG. By the last proposition, we shall have

$$\text{Solid } AG : \text{Solid } AK :: ABCD : AMNO.$$

But the two parallelopipedons AK, AZ, having the same base, $AMNO$, are to each other as their altitudes, AE, AX (Theo. XII.); hence we have

$$\text{Solid } AK : \text{Solid } AZ :: AE : AX.$$

Multiplying together the corresponding terms of these proportions, and omitting, in the result, the common factor *Solid* AK, we shall have

$$\text{Solid } AG : \text{Solid } AZ :: ABCD \times AE : AMNO \times AX.$$

Hence any two rectangular parallelopipedons are to each other as the products of their bases by their altitudes.

298. *Scholium* 1. We are consequently authorized to assume, as the measure of a rectangular parallelopipedon, the product of its base by its altitude; in other words, *the product of its three dimensions.* But by the product of two or more lines is always meant the product of the numbers which represent them; those numbers themselves being determined by the particular linear unit, which may be assumed as the standard. It is necessary, therefore, in comparing magnitudes, that the measuring unit be the same for each of the magnitudes compared.

299. *Scholium* 2. The measured magnitude of a solid, or volume, is called its *volume*, *solidity*, or *solid contents.* We assume as the *unit of volume*, or *solidity*, the cube, each of whose edges is the linear unit, and each of whose faces is the unit of surface.

300. *Cor.* 1. The volume of any rectangular parallelopipedon is the product of its base by its altitude, or the product of its three dimensions.

301. *Cor.* 2. The volume of any cube is the product of its three equal dimensions, or the cube of the number denoting its edges.

THEOREM XV.

302. *The volume of a parallelopipedon, and of any other prism, is equal to the product of its base by its altitude.*

First. Any parellopipedon is equivalent to a rectangular parallelopipedon having the same altitude and an equivalent base (Theo. XI. Cor.). But the volume of a rectangular parallelopipedon is equal to the product of its base by its altitude (Theo. XIV. Cor. 1); therefore the volume of any parallelopipedon is equal to the product of its base by its altitude.

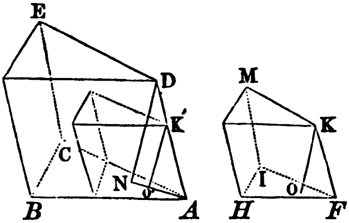
Second. Any parallelopipedon may be divided into two equal triangular prisms of the same altitude, by a plane passing through its opposite diagonal edges. But the volume of the parallelopipedon is equal to the product of its base by its altitude; hence, that of the triangular prism is also equal to the product of its base, or half that of the parallelopipedon, by its altitude.

Third. Any prism may be divided into as many triangular prisms of the same altitude as there are triangles in the polygon taken for a base. But the volume of each triangular prism is equal to the product of its base by its altitude; and, since the altitude is the same in each, it follows that the sum of all these prisms is equal to the sum of all the triangles taken as bases multiplied by the common altitude.

Hence the volume of any prism is equal to the product of its base by its altitude.

THEOREM XVI.

303. *Similar prisms are to each other as the cubes of their homologous edges.*



Let $ABC-E$, $FHI-M$ be two similar prisms; these prisms are to each other as the cubes of their homologous edges, AB and FH.

For, from D and K, homologous angles of the two prisms, draw the perpendiculars DN, KO, to the bases ABC, FHI. Take AK' equal to FK, and join AN. Draw $K'O'$ perpendicular to AN in the plane AND, and $K'O'$ will be perpendicular to the plane ABC, and equal to KO, the altitude of the prism $FHI-M$. For, conceive the triedral angles A and F to be applied the one to the other; the planes containing them, and therefore the perpendiculars $K'O'$, KO, will coincide.

Now, since the bases ABC, FHI are similar, we have (Theo. XIX. Bk. IV.),

$$\textit{Base } ABC : \textit{Base } FHI :: \overline{AB}^2 : \overline{FH}^2;$$

and, because of the similar triangles DAN, KFO, and of the similar parallelograms DB, KH, we have

$$DN : KO :: DA : KF :: AB : FH.$$

Hence, multiplying together the corresponding terms of these proportions, we have

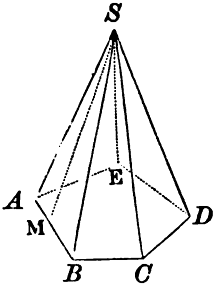
$$\textit{Base } ABC \times DN : \textit{Base } FHI \times KO : \overline{AB}^3 : \overline{FH}^3.$$

But the product of the base by the altitude is equal to the solidity of a prism (Theo. XV.); hence

$$\textit{Prism } ABC-E : \textit{Prism } FHI-M :: \overline{AB}^3 : \overline{FH}^3.$$

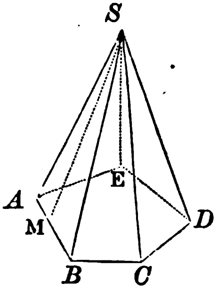
THEOREM XVII.

304. *The convex surface of a right pyramid is equal to the perimeter of its base, multiplied by half the slant height.*



Let $ABCDE-S$ be a right pyramid, and SM its slant height; then the convex surface is equal to the perimeter $AB+BC+CD+DE+EA$ multiplied by $\frac{1}{2}SM$.

The triangles SAB, SBC, SCD, &c. are all equal; for the sides AB, BC, CD, &c. are equal (267), and the sides SA, SB, SC, &c., being oblique lines meeting the base at equal distances from a perpendicular let fall from the vertex S to the center of the base, are also equal (Theo. II.). Hence these triangles are all equal (Theo. XIII. Bk. I.); and the altitude of each is equal to the slant height SM. But the area of a triangle is equal to the product of its base multiplied by half its altitude (Theo. VI. Bk. IV.). Hence the areas of the triangles SAB, SBC, SCD, &c. are equal to the sum of the bases AB, BC,



C D, &c. multiplied by half the common altitude, *S M*; that is, the convex surface of the pyramid is equal to the perimeter of the base multiplied by half the slant height.

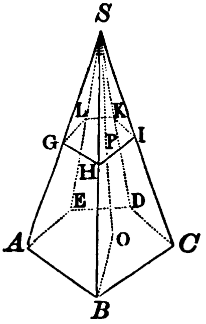
305. *Cor.* The lateral faces of a right pyramid are equal isosceles triangles, having for their bases the sides of the base of the pyramid.

Theorem XVIII

306. ***If a pyramid be cut by a plane parallel to its base,*** —

1*st.* ***The edges and the altitude will be divided proportionally.***

2*d.* ***The section will be a polygon similar to the base.***



Let the pyramid *A B C D E–S*, whose altitude is *S O*, be cut by a plane, *G H I K L*, parallel to its base; then will the edges *S A*, *S B*, *S C*, &c., with the altitude *S O*, be divided proportionally; and the section *G H I K L* will be similar to the base *A B C D E*.

First. Since the planes *A B C*, *G H I* are parallel, their intersections *A B*, *G H*, by the third plane *S A B*, are parallel (Theo. III.); hence the triangles *S A B*, *S G H* are similar (Theo. XVI. Bk. IV.), and we have

$$SA : SG :: SB : SH.$$

For the same reason, we have

$$SB : SH :: SC : SI;$$

and so on. Hence all the edges, *S A*, *S B*, *S C*, &c., are cut proportionally in *G*, *H*, *I*, &c. The altitude *S O* is likewise cut in the same proportion, at the point *P*; for *B O* and *H P* are parallel; therefore we have

$$SO : SP :: SB : SH.$$

Secondly. Since *G H* is parallel to *A B*, *H I* to *B C*, *I K* to *C D*, &c., the angle *G H I* is equal to *A B C*, the angle

HIK to BCD, and so on (Theo. VI.). Also, by reason of the similar triangles SAB, SGH, we have

$$AB : GH :: SB : SH;$$

and by reason of the similar triangles SBC, SHI, we have

$$SB : SH :: BC : HI;$$

hence, on account of the common ratio $SB : SH$,

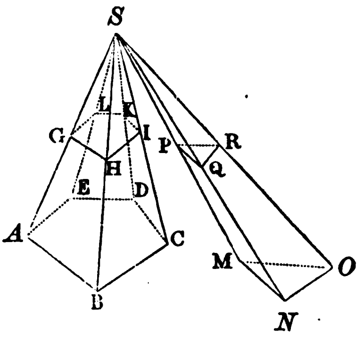
$$AB : GH :: BC : HI.$$

For a like reason, we have

$$BC : HI :: CD : IK,$$

and so on. Hence the polygons $ABCDE$, $GHIKL$ have their angles equal, each to each, and their homologous sides proportional; hence they are similar.

307. *Cor.* 1. ***If two pyramids have the same altitude, and their bases in the same plane, their sections made by a plane parallel to the plane of their bases are to each other as their bases.***



Let $ABCDE-S$, $MNO-S$ be two pyramids, having the same altitude, and their bases in the same plane; and let $GHIKL$, PQR be sections made by a plane parallel to the plane of their bases; then these sections are to each other as the bases $ABCDE$, MNO.

For, the two polygons $ABCDE$, $GHIKL$ being similar, their surfaces are as the squares of the homologous sides AB, GH (Theo. XXI. Bk. IV.). But

$$AB : GH :: SA : SG.$$

Hence, $ABCDE : GHIKL :: \overline{SA}^2 : \overline{SG}^2.$

For the same reason,

$$MNO : PQR :: \overline{SM}^2 : \overline{SP}^2.$$

But since $GHIKL$ and PQR are in the same plane, we have also (Theo. VIII.),

$$SA : SG :: SM : SP :$$

hence,

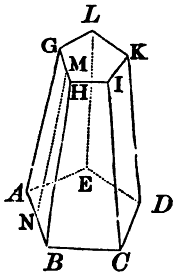
$$ABCDE : GHIKL :: MNO : PQR,$$

therefore the sections $GHIKL$, PQR are to each other as the bases $ABCDE$, MNO.

308. *Cor.* 2. If the bases $ABCDE$, MNO are equivalent, any sections, $GHIKL$, PQR, made at equal distances from those bases, are likewise equivalent.

THEOREM XIX.

309. ***The convex surface of a frustum of a right pyramid is equal to half the sum of the perimeters of its two bases, multiplied by its slant hight.***

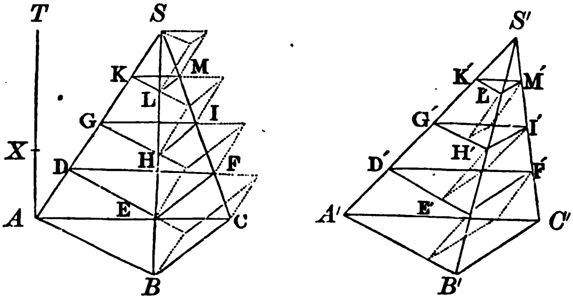


Let $ABCDE-L$ be the frustum of a right pyramid, and MN its slant hight; then the convex surface is equal to the sum of the perimeters of the two bases $ABCDE$, $GIHKL$, multiplied by half of MN.

For the upper base $GHIKL$ is similar to the base $ABCDE$ (Theo. XVIII.), and $ABCDE$ is a regular polygon (267); hence the sides GH, HI, IK, KL, and LG are all equal to each other. The angles GAB, ABH, HBC, &c. are equal (Theo. XVII. Cor.), and the edges AG, BH, CI, &c. are also equal (Theo. XVIII.); therefore the faces AH, BI, CK, &c. are all equal trapezoids (23), having a common altitude, MN, the slant hight of the frustum. But the area of either trapezoid, as AH, is equal to $\frac{1}{2}(AB + GH) \times MN$ (Theo. VII. Bk. IV.); hence the areas of all the trapezoids, or the convex surface of frustum, are equal to half the sum of the perimeters of the two bases multiplied by the slant hight.

THEOREM XX.

310. *Triangular pyramids, having equivalent bases and the same altitude, are equivalent.*



Let $A\,B\,C-S$, $A'\,B'\,C'-S'$ be two triangular pyramids, having equivalent bases, $A\,B\,C$, $A'\,B'\,C'$, situated in the same plane; and let them have the same altitude, $A\,T$; then these pyramids are equivalent.

For, if the two pyramids are not equivalent, let $A'\,B'\,C'-S'$ be the smaller, and suppose $A\,X$ to be the altitude of a prism, which, having $A\,B\,C$ for its base, is equal to their difference.

Divide the altitude $A\,T$ into equal parts, each less than $A\,X$; through each point of division pass a plane parallel to the plane of the base, thus forming corresponding sections in the two pyramids equivalent each to each, namely, $D\,E\,F$ to $D'\,E'\,F'$, $G\,H\,I$ to $G'\,H'\,I'$, &c.

Upon the triangles $A\,B\,C$, $D\,E\,F$, $G\,H\,I$, &c., taken as bases, construct exterior prisms, having for edges the parts $A\,D$, $D\,G$, $G\,K$, &c. of the edge $S\,A$; in like manner, on the bases $D'\,E'\,F'$, $G'\,H'\,I'$, &c. in the second pyramid, construct interior prisms, having for edges the corresponding parts of $S'\,A'$. It is plain that the sum of all the exterior prisms of the pyramid $A\,B\,C-S$ is greater than this pyramid; and also that the sum of all the interior prisms of the pyramid $A'\,B'\,C'-S'$ is less than this pyramid. Hence, the difference between the sum of all the exterior prisms and the sum of all the interior ones, must be greater than the difference between the two pyramids themselves.

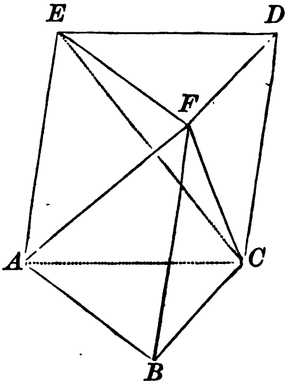
Now, beginning with the bases $A\,B\,C$, $A'\,B'\,C'$, the second exterior prism, $D\,E\,F-G$, is equivalent to the first interior prism, $D'\,E'\,F'-A'$, since they have equal altitudes, and their bases, $D\,E\,F$, $D'\,E'\,F'$, are equivalent. For a like reason, the third exterior prism, $G\,H\,I-K$, and the second interior prism $G'\,H'\,I'-D'$, are equivalent, and so on to the last in each series. Hence, all the exterior prisms of the pyramid $A\,B\,C-S$, excepting the first prism, $A\,B\,C-D$, have equivalent corresponding ones in the interior prisms of the pyramid $A'\,B'\,C'-S'$. Therefore the prism $A\,B\,C-D$ is the difference between the sum of all the exterior prisms of the pyramid $A\,B\,C-S$, and the sum of the interior prisms of the pyramid $A'\,B'\,C'-S'$. But the difference between these two sets of prisms has been proved to be greater than that of the two pyramids, which latter difference we supposed to be equal to the prism $A\,B\,C-X$. Hence, the prism $A\,B\,C-D$ must be greater than the prism $A\,B\,C-X$, which is impossible, since they have the same base, $A\,B\,C$, and the altitude of the first is less than $A\,X$, the altitude of the second. Hence, the supposed inequality between the two pyramids cannot exist; therefore the two pyramids $A\,B\,C-S$, $A'\,B\,C'-S'$, having the same altitude and equivalent bases, are themselves equivalent.

THEOREM XXI.

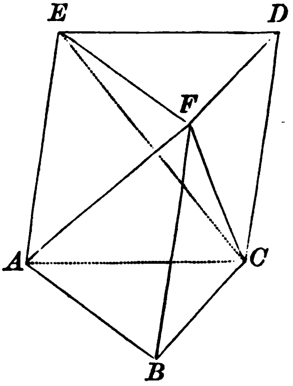
311. *Every triangular pyramid is a third part of a triangular prism having the same base and the same altitude.*

Let $A\,B\,C-F$ be a triangular pyramid, and $A\,B\,C-D\,E\,F$ a triangular prism of the same base and the same altitude; then the pyramid is one third of the prism.

Cut off the pyramid $A\,B\,C-F$ from the prism, by the plane $F\,A\,C$; there will remain the solid $A\,C\,D\,E-F$, which may be considered as a quadrangular pyramid, whose vertex is F, and



whose base is the parallelogram *A C D E*. Draw the diagonal *C E*, and pass the plane *F C E*, which will cut the quadrangular pyramid into two triangular ones, *A C E*-*F*, *E D C*-*F*. These two triangular pyramids have for their common altitude the perpendicular let fall from *F* on the plane *A C D E;* they have equal bases, since the triangles *A C E*, *C D E* are halves of the same parallelogram; hence the two pyramids *A C E*-*F*, *C D E*-*F* are equivalent (Theo. XX.) But the pyramid *C D E*-*F* and the pyramid *A B C*-*F*, have equal bases, *A B C*, *D E F;* they have also the same altitude, namely, the distance between the parallel planes *A B C*, *D E F;* hence the two pyramids are equivalent. Now, the pyramid *C D E*-*F* has been proved equivalent to *A C E*-*F;* hence the three pyramids *A B C*-*F*, *C D E*-*F*, *A C E*-*F*, which compose the whole prism *A B C*-*D E F*, are all equivalent; therefore, either pyramid, as *A B C*-*F*, is the third part of the prism, which has the same base and the same altitude.

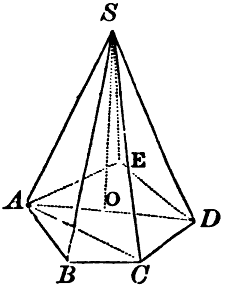


312. *Cor.* 1. Every triangular prism may be divided into three equivalent triangular pyramids.

313. *Cor.* 2. The solidity of a triangular pyramid is equal to a third part of the product of its base by its altitude.

THEOREM XXII.

314. *The volume of every pyramid is equal to the product of its base by one third of its altitude.*



Let $ABCD-S$ be any pyramid, whose base is $ABCDE$, and altitude SO; then its solidity is equal to $ABCDE \times \frac{1}{3} SO$.

Draw the diagonals AC, AD, and pass the planes SAC, SAD through these diagonals and the vertex S; the polygonal pyramid $ABCDE-S$ will be divided into several triangular pyramids, all having the same altitude, SO. But each of these pyramids is measured by the product of its base, BAC, CAD, DAE, by a third part of its altitude, SO (Theo. XXI. Cor. 2); hence, the sum of these triangular pyramids, or the polygonal pyramid $ABCDE-S$, will be measured by the sum of the triangles BAC, CAD, DAE, or the polygon $ABCDE$, multiplied by $\frac{1}{3}$ of SO; hence, every pyramid is measured by the product of its base by one third of its altitude.

315. *Cor.* 1. Every pyramid is the third part of the prism which has the same base and the same altitude.

316. *Cor.* 2. Pyramids having the same altitude are to each other as their bases.

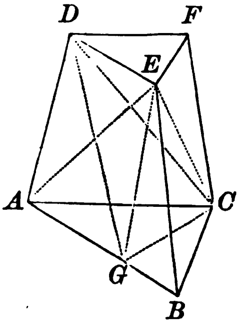
317. *Cor.* 3. Pyramids having the same base, or equivalent bases, are to each other as their altitudes.

318. *Cor.* 4. Pyramids are to each other as the products of their bases by their altitudes.

319. *Scholium.* The volume of any polyedron may be found by dividing it into pyramids, by passing planes through its vertices.

THEOREM XXIII.

320. *A frustum of a pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*



First. Let $ABC-DEF$ be the frustum of a pyramid, whose base is a triangle. Pass a plane through the points A, E, C; it cuts off the triangular pyramid $ABC-E$, whose altitude is that of the frustum, and whose base, ABC, is the lower base of the frustum. Pass another plane through the points DEC; it cuts off the triangular pyramid $DEF-C$, whose altitude is that of the frustum, and whose base, DEF, is the upper base of the frustum.

There now remains of the frustum the pyramid $ACD-E$. Draw EG parallel to AD; join CG and DG. Then, since EG is parallel to AD, a line in the plane ACD, it is parallel to the plane ACD (Theo. III.); and the pyramid $ACD-E$ is equivalent to the pyramid $ACD-G$, since they have the same base, ACD, and their vertices, E and G, lie in the same straight line parallel to the common base. But the pyramid $ACD-G$ is the same as the pyramid $AGC-D$, whose altitude is that of the frustum, and whose base, AGC, as will be proved, is a mean proportional between the bases ABC and DEF.

The two triangles AGC, DEF have the angles A and D equal to each other (Theo. VI.); hence we have (Theo. XVIII. Bk. IV.),

$$AGC : DEF :: AG \times AC : DE \times DF;$$

but since AG is equal to DE,

$$AGC : DEF :: AC : DF.$$

We have also (Theo. VI. Cor., Bk. IV.),

$$ABC : AGC :: AB : AG \text{ or } DE.$$

But the similar triangles ABC, DEF give

$$AB : DE :: AC : DF;$$

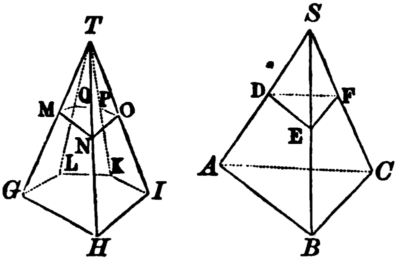
hence (Theo. IX. Bk. II.),

$$ABC : AGC :: AGC : DEF;$$

that is, the base AGC is a mean proportional between the bases ABC, DEF of the frustum.

Secondly. Let $GHIKL-MNOPQ$ be the frustum of a pyramid, whose base is any polygon.

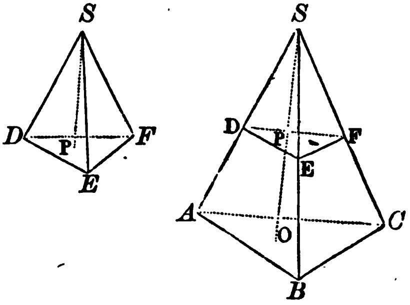
Let $ABC-S$ be a triangular pyramid having the same altitude, and an equivalent base, with any polygonal pyramid, $GHIKL-T$; these pyramids are equivalent (Theo. XXII. Cor. 3).



The bases of the two pyramids may be regarded as situated in the same plane, in which case the plane $MNOPQ$ produced will form in the triangular pyramid a section, DEF, at the same distance above the common plane of the bases; and therefore the section DEF will be to the section $MNOPQ$ as the base ABC is to the base $GHIKL$ (Theo. XVIII. Cor. 1); and since the bases are equivalent, the sections will be so likewise. Hence, the pyramids $MNOPQ-T$, $DEF-S$, having the same altitude and equivalent bases, are equivalent. For the same reason, the entire pyramids $GHIKL-T$, $ABC-S$ are equivalent; consequently, the frustums $GHIKL-MNOPQ$, $ABC-DEF$, are equivalent. But the frustum $ABC-DEF$ has been shown to be equivalent to the sum of three pyramids having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them. Hence the proposition is true of the frustum of any pyramid.

THEOREM XXIV.

321. *Similar pyramids are to each other as the cubes of their homologous edges.*



Let $A\,B\,C - S$ and $D\,E\,F - S$ be two similar pyramids; these pyramids are to each other as the cubes of their homologous edges $A\,B$ and $D\,E$, or $B\,C$ and $E\,F$, &c.

For, the two pyramids being similar, the homologous polyedral angles at the vertices are equal (274); hence the smaller pyramid may be so applied to the larger, that the polyedral angle S shall be common to both.

In that case, the bases $A\,B\,C$, $D\,E\,F$ will be parallel; for, since the homologous faces are similar, the angle $S\,D\,E$ is equal to $S\,A\,B$, and $S\,E\,F$ to $S\,B\,C$; hence the plane $A\,B\,C$ is parallel to the plane $D\,E\,F$ (Theo. VI.). Then let $S\,O$ be drawn from the vertex S perpendicular to the plane $A\,B\,C$, and let P be the point where this perpendicular meets the plane $D\,E\,F$. From what has already been shown (Theo. XVIII.), we shall have

$$S\,O : S\,P :: S\,A : S\,D :: A\,B : D\,E;$$

and consequently,

$$\tfrac{1}{3}\,S\,O : \tfrac{1}{3}\,S\,P :: A\,B : D\,E.$$

But the bases $A\,B\,C$, $D\,E\,F$ are similar; hence (Theo. XIX. Bk. IV.),

$$A\,B\,C : D\,E\,F :: \overline{A\,B}^2 : \overline{D\,E}^2.$$

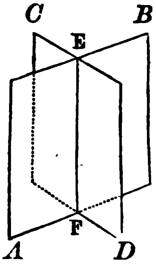
Multiplying together the corresponding terms of these two proportions, we have

$$ABC \times \tfrac{1}{3} SO : DEF \times \tfrac{1}{3} SP :: \overline{AB}^3 : \overline{DE}^3.$$

Now, $ABC \times \frac{1}{3} SO$ represents the volume of the pyramid $ABC-S$, and $DEF \times \frac{1}{3} SP$ that of the pyramid $DEF-S$ (Theo. XXII.); hence two similar pyramids are to each other as the cubes of their homologous edges.

EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

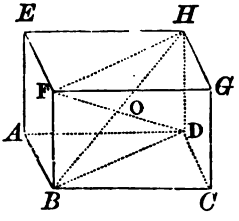


322. 1. A straight line cannot be partly in a plane, and partly out of it.

2. If two planes cut each other, their common section is a straight line.

3. If two planes are perpendicular to the same straight line they are parallel to each other.

4. In every parallelopipedon the opposite faces are equal and parallel.



5. The diagonals of every parallelopipedon bisect each other.

6. The volume of a triangular prism is equal to the product of the area of either of its rectangular sides as a base multiplied by half its altitude on that base.

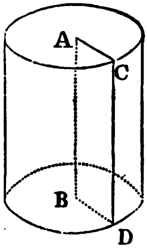
7. All prisms of equal bases and altitudes are equal in volume, whatever be the figure of their bases.

BOOK VI.

THE THREE ROUND BODIES.

DEFINITIONS

323. A **Cylinder** is a volume which may be described by the revolution of a rectangle turning about one of its sides, which remains immovable; as the solid described by the rectangle *A B C D* revolving about its side *A B*.

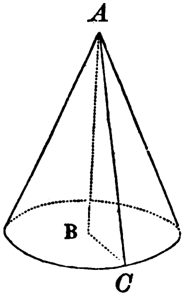


324. The **Bases** of the cylinder are the circles described by the sides, *A C*, *B D*, of the revolving rectangle, which are adjacent to the immovable side, *A B*.

The **Axis** of the cylinder is the straight line joining the centers of its two bases; as the immovable line *A B*.

The **Convex Surface** of the cylinder is described by the side *C D* of the rectangle, opposite to the axis *A B*.

325. A **Cone** is a solid which may be described by the revolution of a right-angled triangle turning about one of its perpendicular sides, which remains immovable; as the solid described by the right-angled triangle *A B C* revolving about its perpendicular side *A B*.



The **Base** of the cone is the circle described by the revolution of the side *B C*, which is perpendicular to the immovable side.

The **Convex Surface** of a cone is described by the hypothenuse, *A C*, of the revolving triangle.

The **Vertex** of the cone is the point A, where the hypothenuse meets the immovable side.

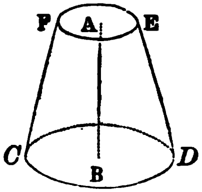
The **Axis** of the cone is the straight line joining the vertex to the center of the base; as the line $A\,B$.

The **Altitude** of a cone is a line drawn from the vertex perpendicular to the base; and is the same as the axis, $A\,B$.

The **Slant Hight,** or **Side,** of a cone, is a straight line drawn from the vertex to the circumference of the base; as the lins $A\,C$.

326. The **Frustum** of a cone is the part of a cone included between the base and a plane parallel to the base; as the solid $C\,D-F$.

The **Axis,** or **Altitude,** of the frustum, is the perpendicular line $A\,B$ included between the two bases; and the **Slant Hight,** or **Side,** is that portion of the slant hight of the cone which lies between the bases; as $F\,C$.

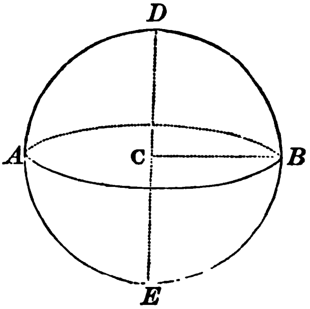


327. **Similar Cylinders,** or **Cones,** are those whose axes are to each other as the radii, or diameters, of their bases.

328. A **Sphere** is a solid, or volume, bounded by a curved surface, all points of which are equally distant from a point within, called the center.

The sphere may be conceived to be formed by the revolution of a semicircle, $D\,A\,E$, about its diameter, $D\,E$, which remains fixed.

329. The **Radius** of a sphere is a straight line drawn from the center to any point in the surface, as the line $C\,B$.



The **Diameter,** or **Axis,** of a sphere is a line passing through the center, and terminated both ways by the surface, as the line $D\,E$.

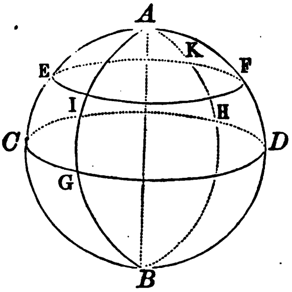
Hence, all the radii of a sphere are equal; and all the diamters are equal, and each is double the radius.

330. **A Great Circle** of the sphere is a section made by a plane passing through the center, and having the center of the sphere for its center; as the section $A B$, whose center is C.

A Small Circle of the sphere is any section made by a plane not passing through the center.

A Plane is Tangent to a sphere, when it meets the sphere in but one point, however far it may be produced.

331. **A Spherical Polygon** is a portion of the surface of a sphere bounded by several arcs of great circles.



332. **A Zone** is a portion of the surface of a sphere cut off by a plane, or comprehended between two parallel planes; as $E I F K - A$, or $C G D H - E I F K$.

333. **A Spherical Segment** is a portion of the sphere cut off by a plane, or comprehended between two parallel planes.

334. **The Altitude of a Zone** or of a SPHERICAL SEGMENT is the perpendicular distance between the two parallel planes which comprehend the zone or segment.

In case the zone or segment is a portion of the sphere cut off, one of the planes is a tangent to the sphere.

335. **A Spherical Sector** is a solid described by the revolution of a circular sector, in the same manner as the semicircle of which it is a part, by revolving round its diameter, describes a sphere.

336. **A Spherical Pyramid** is a portion of the sphere comprehended between the planes of a polyedral angle whose vertex is the center.

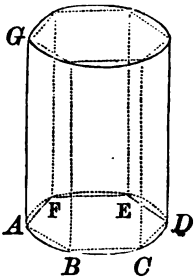
The *base* of the pyramid is the spherical polygon intercepted by the same planes.

337. The cylinder, cone, and sphere are termed the **Three Round Bodies of Elementary Geometry.**

THEOREM I.

338. *The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.*

Let $A\,B\,C\,D\,E\,F-G$ be a cylinder, whose circumference is the circle $A\,B\,C\,D\,E\,F$, and whose altitude is the line $A\,G$; then its convex surface is equal to $A\,B\,C\,D\,E\,F$ multiplied by $A\,G$.



In the base of the cylinder inscribe any regular polygon, $A\,B\,C\,D\,E\,F$, and on this polygon construct a right prism of the same altitude with the cylinder. The prism will be inscribed in the convex surface of the cylinder. The convex surface of this prism is equal to the perimeter of its base multiplied by its altitude, $A\,G$ (Theo. IX. Bk. V.).

Conceive now the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will then be equal to the circumference of the circle $A\,B\,C\,D\,E\,F$ (Theo. XXIX. Cor., Bk. IV.); and thus the convex surface of the prism will coincide with the convex surface of the cylinder. But the convex surface of the prism is always equal to the perimeter of its base multiplied by its altitude; hence, the convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude.

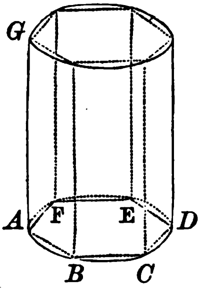
339. *Cor.* 1. If two cylinders have the same altitude, their convex surfaces are to each other as the circumferences of their bases.

340. *Cor.* 2. If H represent the altitude of a cylinder, and R the radius of its base, then we shall have the circumference of the base represented by $2R \times \pi$ (Theo. XXXII. Cor. 3, Bk. IV.), and the convex surface of the cylinder by $2R \times \pi \times H$.

THEOREM II.

341. *The volume of a cylinder is equal to the product of its base by its altitude.*

Let $ABCDEF-G$ be a cylinder whose base is the circle $ABCDEF$, and whose altitude is the line AG; then its volume is equal to the product $ABCDEF$ by AG.



In the base of the cylinder inscribe any regular polygon, $ABCDEF$, and on this polygon construct a right prism of the same altitude with the cylinder. The prism will be inscribed in the convex surface of the cylinder. The solid contents of this prism are equal to the product of its base by its altitude (Theo. XV. Bk. V.).

Conceive now the number of the sides of the polygon to be indefinitely increased, until its perimeter coincides with the circumference of the circle $ABCDEF$ (Theo. XXIX. Cor., Bk. IV.), and the volume of the prism will equal that of the cylinder. But the volume of the prism will still be equal to the product of its base by its altitude; hence the solid contents of the cylinder are equal to the product of its base by its altitude.

342. *Cor.* 1. Cylinders of the same altitude are to each other as their bases; and cylinders of equal bases are to each other as their altitudes.

343. *Cor.* 2. Similar cylinders are to each other as the cubes of their altitudes, or as the cubes of the diameters of their bases. For the bases are as the squares of their radii

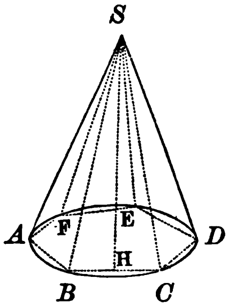
(Theo. XXX. Bk. IV.), and the cylinders being similar, the radii of their bases are to each other as their altitudes (327); therefore the bases are as the squares of the altitudes; hence, the products of the bases by the altitudes, or the cylinders themselves, are as the cubes of the altitudes.

344. *Cor.* 3. If the altitude of a cylinder be represented by H, and the area of its base by $R^2 \times \pi$ (Theo. XXXII. Cor. 2, Bk. IV.), the solid contents of the cylinder will be represented by $R^2 \times \pi \times H$.

THEOREM III.

345. *The convex surface of a cone is equal to the circumference of the base multiplied by half the slant hight.*

Let $A\,B\,C\,D\,E\,F - S$ be a cone whose base is the circle $A\,B\,C\,D\,E\,F$, and whose slant hight is the line $S\,A$; then its convex surface is equal to $A\,B\,C\,D\,E\,F$ multiplied by $\frac{1}{2}\,S\,A$.



In the base of the cone inscribe any regular polygon, $A\,B\,C\,D\,E\,F$, and on this polygon construct a regular pyramid having the same vertex, S, with the cone. Then a right pyramid will be inscribed in the cone.

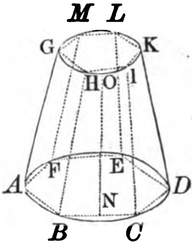
From S draw $S\,H$ perpendicular to $B\,C$, a side of the polygon. The convex surface of the pyramid is equal to the perimeter of its base multiplied by half its slant hight $S\,H$ (Theo. XVII. Bk. V.). Conceive now the arcs subtending the sides of the polygon to be continually bisected, until a polygon is formed having an indefinite number of sides; its perimeter will equal the circumference of the circle $A\,B\,C\,D\,E\,F$; its slant hight, $S\,H$, will equal that of the cone, and its convex surface coincide with the convex surface of the cone. But the convex surface of every right pyramid is equal to the perimeter of its base, multiplied by half the slant hight;

hence the convex surface of the cone is equal to the circumference of its base multiplied by half its slant hight.

346. *Cor.* If SA represent the slant hight of a cone, and R the radius of the base, then, since the circumference of the base is represented by $2R \times \pi$ (Theo. XXXII. Cor. 3, Bk. IV.), the convex surface of the cone will be represented by $2R \times \pi \times \frac{1}{2} SA$, equal to $\pi \times R \times SA$.

Theorem IV.

347. *The convex surface of a frustum of a cone is equal to half the sum of the circumference of the two bases multiplied by its slant hight.*

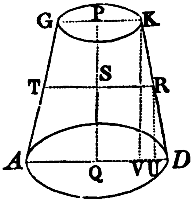


Let $ABCDEF-M$ be the frustum of a cone, and AG its slant hight; then the convex surface is equal to half the sum of the circumferences of the two bases $ABCDEF$, $GHIKLM$, multiplied by AG.

For, inscribe in the bases of the frustum two regular polygons of the same number of sides, having their sides parallel, each to each. Draw the straight lines AG, BH, CI, &c., joining the vertices of the corresponding angles, and these lines will be the edges of the frustum of a pyramid inscribed in the frustum of the cone. The convex surface of the frustum of the pyramid is equal to half the sum of the perimeters of the two bases multiplied by its slant hight, ON (Theo. XIX. Bk. V.).

Conceive now the number of sides of the inscribed polygons to be indefinitely increased; the perimeters of the polygons will then coincide with the circumferences of the circles $ABCDEF$, $GHIKLM$; and the slant hight, ON, of the frustum of the pyramid, will equal the slant hight, AG, of the frustum of the cone; and the surfaces of the two frustums will coincide.

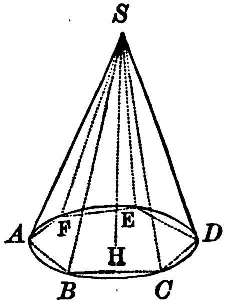
But the convex surface of every frustum of a right pyramid is equal to half the sum of the perimeters of its two bases, multiplied by its slant hight; hence, the convex surface of the frustum of the cone is equal to half the sum of the circumference of its two bases multiplied by half its slant hight.



348. *Cor.* Through R, the middle point of the side KD, draw the diameter RST, parallel to the diameter AQD, and the straight lines RU, KV, parallel to the axis PQ. Then, since DR is equal to RK, DU is equal to UV (Theo. X. Cor. 2, Bk. IV.); hence, the radius SR is equal to half the sum of the radii QD, PK. But the circumferences of circles being to each other as their radii (Theo. XXX. Bk. IV.), the circumference of the section of which SR is the radius is equal to half the sum of the circumferences of which QD, PK are the radii; hence, the convex surface of a frustum of a cone is equal to the slant hight multiplied by the circumference of a section at equal distances between the two bases.

THEOREM V.

349. *The volume of a cone is equal to the product of its base by one third of its altitude.*



Let $ABCDEF-S$ be a cone, whose base is $ABCDEF$, and altitude SH; then its solidity is equal to $ABCDEF \times \frac{1}{3} SH$.

In the base of the cone inscribe any regular polygon, $ABCDEF$, and on this polygon construct a regular pyramid, having the same vertex, S, with the cone. Then a right pyramid will be inscribed in the cone; and its volume will be equal to the product of its base by one third of its altitude (Theo. XXII. Bk. V.).

Conceive, now, the number of sides of the polygon to be indefinitely increased, and its perimeter will become equal to the circumference of the cone, and the pyramid will exactly coincide with the cone. But the volume of every right pyramid is equal to the product of the base by one third of its altitude; hence the volume of a cone is equal to the product of its base by one third of its altitude.

350. *Cor.* 1. A cone is the third of a cylinder having the same base and the same altitude; hence it follows,—

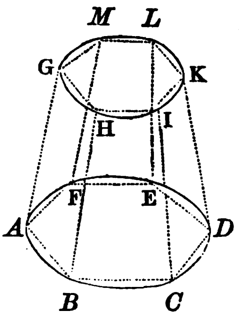
1. That cones of equal altitude are to each other as their bases;
2. That cones of equal bases are to each other as their altitudes;
3. That similar cones are as the cubes of the diameters of their bases, or as the cubes of their altitudes.

351. *Cor.* 2. If the altitude of a cone be represented by H, and the radius of its base by R, the volume of the cone will be represented by

$$R^2 \times \pi \times \tfrac{1}{3} H, \text{ or } \tfrac{1}{3} \pi \times R^2 \times H.$$

THEOREM VI.

352. *The volume of the frustum of a cone is equivalent to the sum of three cones, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.*



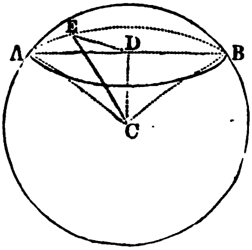
Let $A\,B\,C\,D\,E\,F-M$ be the frustum of a cone; then will its volume be equivalent to the sum of three cones having the same altitude as the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

For, inscribe in the two bases of the frustum two regular polygons having the same number of sides, and having their sides parallel, each to each. Let the vertices of the corresponding angles be joined by the straight lines BH, CI, &c., and there is inscribed in the frustum of the cone the frustum of a regular pyramid. The volume of the frustum of this pyramid is equivalent to the sum of three pyramids, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

Conceive now the number of the sides of the polygons to be indefinitely increased; and the bases of the frustum of the pyramid will equal the bases of the frustum of the cone; and the two frustums will coincide. Hence the frustum of a cone is equivalent to the sum of three cones, having for their common altitude the altitude of the frustum, and whose bases are the two bases of the frustum, and a mean proportional between them.

THEOREM VII.

353. *Every section of a sphere made by a plane is a circle.*



Let ABE be a section made by a plane in the sphere whose center is C. From the center, C, draw CD perpendicular to the plane ABE; and draw the lines CA, CB, CE, to different points of the curve ABE, which bounds the section.

The oblique lines CA, CB, CE are equal, being radii of the sphere; therefore they are equally distant from the perpendicular, CD (Theo. II. Cor., Bk. V.). Hence the lines DA, DB, DE, and, in like manner, all the lines drawn from D to the boundary of the section, are equal; and therefore the section ABE is a circle whose center is D.

354. *Cor.* 1. If the section passes through the center of the sphere, its radius will be the radius of the sphere; hence all great circles are equal.

355. *Cor.* 2. Two great circles always bisect each other. For, since the two circles have the same center, their common intersection, passing through the center, must be a common diameter bisecting both circles.

356. *Cor.* 3. Every great circle divides the sphere and its surface into two equal parts. For if the two hemispheres were separated, and afterwards placed on the common base, with their convexities turned the same way, the two surfaces would exactly coincide.

357. *Cor.* 4. The center of a small circle, and that of the sphere, are in a straight line perpendicular to the plane of the small circle.

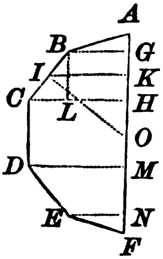
358. *Cor.* 5. Small circles are less according to their distance from the center; for, the greater the distance CD, the smaller the chord AB, the diameter of the small circle ABE.

359. *Cor.* 6. The arc of a great circle may be made to pass through any two points on the surface of a sphere; for the two given points and the center of the sphere determine the position of a plane. If, however, the two given points be the extremities of a diameter, these two points and the center would be in a straight line, and any number of great circles may be made to pass through the two given points.

THEOREM VIII.

360. *If any regular semi-polygon be revolved about a line passing through the center and the vertices of opposite angles, the surface described will be equal to the product of its axis by the circumference of its inscribed circle.*

Let the regular semi-polygon $ABCDEF$ be revolved about AF as an axis; then the surface described by the sides



$A B$, $B C$, $C D$, &c. will equal the product of $A F$ by the inscribed circle.

For, from the vertices B, C, D, E of the semi-polygon, draw $B G$, $C H$, $D M$, $E N$, perpendicular to the axis $A F$; and from the center, O, draw $O I$ perpendicular to one of the sides; also draw $I K$ perpendicular to $A F$, and $B L$ perpendicular to $C H$.

Now, $O I$ is the radius of the inscribed circle (Theo. XXIII. Bk. IV.); and the surface described by the revolution of a side, $B C$, of a regular polygon, is equal to $B C$ multiplied by the circumference, $I K$ (Theo. IV. Cor.).

The two triangles $O I K$, $B C L$, having their sides perpendicular to each other, are similar (Theo. XVI. Bk. IV.); therefore,

$$B C : B L \text{ or } G H :: O I : I K :: \textit{Circ. } O I : \textit{Circ. } I K.$$

Hence (Theo. I. Bk. II.),

$$B C \times \textit{Circ. } I K = G H \times \textit{Circ. } O I;$$

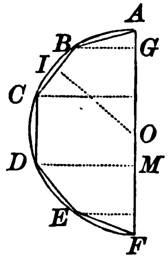
that is, the surface described by $B C$ is equal to the product of the altitude $G H$ by the circumference of the inscribed circle. The same may be shown of each of the other sides; hence, the surface described by all the sides taken together is equal to the product of the sum of the altitudes $A G$, $G H$, $H M$, $M N$, $N F$, by the circ. $O I$, or to the product of the axis $A F$ by the circ. $O I$.

THEOREM IX.

361. *The surface of a sphere is equal to the product of its diameter by the circumference of a great circle.*

Let $A B C D E F$ be a semicircle in which is inscribed any regular semi-polygon; from the center, O, draw $O I$ perpendicular to one of the sides.

If now the semicircle and the semi-polygon be revolved about the axis $A F$, the surface described by the semicircle will be the surface of a sphere (328), and that described by the



semi-polygon will be equal to the product of its axis, *A F*, by the circumference, *O I* (Theo. VIII.); and the same is true, whatever be the number of sides of the polygon.

Conceive the number of sides of the semi-polygon to be made, by continual bisections, indefinitely great; then its perimeter will coincide with the semi-circumference *A B C D E F*, and the perpendicular *O I* will be equal to the radius *O A;* hence, the surface of the sphere is equal to the product of the diameter by the circumference of a great circle.

362. *Cor.* 1. *The surface of a sphere is equal to the area of four of its great circles.*

For the area of a circle is equal to the product of the circumference by half the radius, or one fourth of the diameter (Theo. XXXII. Bk. IV.).

363. *Cor.* 2. *The surface of a zone or segment is equal to the product of its altitude by the circumference of a great circle.*

For the surface described by the sides *B C*, *C D* of the inscribed polygon is equal to the product of the altitude *G M* by the circumference of the inscribed circle *O I*. If, now, the number of the sides of an inscribed polygon be indefinitely increased, its perimeter will equal the circle, and *B C*, *C D* will coincide with the arc *B C D;* consequently, the surface of the zone described by the revolution of *B C D* is equal to the product of its altitude by the circumference of a great circle. In like manner, the same may be proved true of a segment, or a zone having but one base.

364. *Cor.* 3. The surfaces of two zones, or segments upon the same sphere, are to each other as their altitudes; and any zone or segment is to the surface of the sphere as the altitude of that zone or segment is to the diameter.

365. *Cor.* 4. If the radius of a sphere is represented by R, and its diameter by D, its surface will be represented by

$$4\pi \times R^2, \text{ or } \pi \times D^2.$$

366. *Cor.* 5. Hence, the surfaces of spheres are to each other as the squares of their radii or diameters.

367. *Cor.* 6. If the altitude of a zone or segment is represented by H, the surface of a zone or segment will be represented by

$$2\pi \times R \times H, \text{ or } \pi \times D \times H.$$

Theorem X.

368. *The volume of a sphere is equal to the product of its surface by one third of its radius.*

For a sphere may be regarded as composed of an indefinite number of pyramids, each having for its base a part of the surface of the sphere, and for its vertex the center of the sphere; consequently, all these pyramids have the radius of the sphere as their common altitude.

Now, the volume of every pyramid is equal to the product of its base by one third of its altitude (Theo. XXII. Bk. V.); hence, the sum of the volumes of these pyramids is equal to the product of the sum of their bases by one third of their common altitude. But the sum of their bases is the surface of the sphere, and their common altitude its radius; consequently, the volume of the sphere is equal to the product of its surface by one third of it radius.

369. *Cor.* 1. *The volume of a spherical pyramid or sector is equal to the product of the polygon or zone which forms its base, by one third of the radius.*

For the polygon or zone forming the base of the spherical pyramid or sector may be regarded as composed of an indefinite number of planes, each serving as a base to a pyramid, having for its vertex the center of the sphere.

370. *Cor.* 2. Spherical pyramids, or sectors of the same sphere or of equal spheres, are to each other as their bases.

371. *Cor.* 3. A spherical pyramid or sector is to the sphere of which it is a part, as its base is to the surface of the sphere.

372. *Cor.* 4. Hence, spherical sectors upon the same sphere are to each other as the altitudes of the zones forming their bases (Theo. IX. Cor. 3); and any spherical sector is to the sphere as the altitude of the zone forming its base is to the diameter of the sphere.

373. *Cor.* 5. If the radius of a sphere is represented by R, its diameter by D, and its surface by S, its volume will be represented by

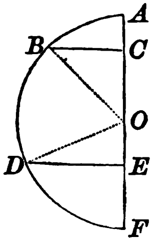
$$S \times \tfrac{1}{3} R = 4\pi \times R^2 \times \tfrac{1}{3} R = \tfrac{4}{3}\pi \times R^3 \text{ or } \tfrac{1}{6}\pi \times D^3.$$

374. *Cor.* 6. Hence, the volumes of spheres are to each other as the cubes of their radii.

375. *Cor.* 7. If the altitude of the zone which forms the base of a sector be represented by H, the volume of the sector will be represented by

$$2\pi \times R \times H \times \tfrac{1}{3} R = \tfrac{2}{3}\pi \times R^2 \times H.$$

376. *Scholium.* The volume of the spherical segment less than a hemisphere, and of one base, formed by the revolution of a portion, $A\,B\,C$, of a semicircle about the radius OA, is equivalent to the volume of the spherical sector formed by $A\,O\,B$, less than the volume of the cone formed by $O\,B\,C$.



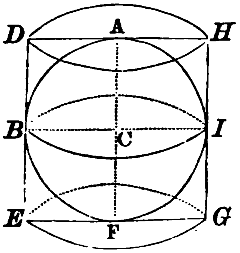
The volume of the spherical segment greater than a hemisphere, and of one base, formed by the revolution of $A\,D\,E$, is equivalent to the volume of the spherical sector formed by $A\,O\,D$, plus the volume of the cone formed by $O\,D\,E$.

The volume of the spherical segment of two bases formed by the revolution of $C\,B\,D\,E$ about the axis $A\,F$, is equivalent to the solidity of the segment formed by $A\,D\,E$, less the volume of the segment formed by $A\,B\,C$.

THEOREM XI.

377. *The surface of a sphere is equivalent to the convex surface of the circumscribed cylinder, and is two thirds of the whole surface of the cylinder; also, the volume of the sphere is two thirds of that of the circumscribed cylinder.*

Let *A B F I* be a great circle of the sphere; *D E G H* the circumscribed square; then, if the semicircle *A B F* and the semi-square *A D E F* be revolved about the diameter *A F*, the semicircle will describe a sphere, and the semi-square a cylinder circumscribing the sphere.



The convex surface of the cylinder is equal to the circumference of its base multiplied by its altitude (Theo. I.). But the base of the cylinder is equal to the great circle of the sphere, its diameter *E G* being equal to the diameter *B I*, and the altitude *D E* is equal to the diameter *A F*; hence, the convex surface of the cylinder is equal to the circumference of the great circle multiplied by its diameter. This measure is the same as that of the surface of the sphere (Theo. IX.); hence, the surface of the sphere is equal to the convex surface of the circumscribed cylinder.

But the surface of the sphere is equal to four great circles of the sphere (Theo. IX. Cor. 1); hence, the convex surface of the cylinder is also equal to four great circles; and adding the two bases, each equal to a great circle, the whole surface of the circumscribed cylinder is equal to six great circles of the sphere; hence, the surface of the sphere is $\frac{4}{6}$, or $\frac{2}{3}$ of the whole surface of the circumscribed sphere.

In the next place, since the base of the circumscribed cylinder is equal to a great circle of the sphere, and its altitude to the diameter, the volume of the cylinder is equal to a great circle multiplied by its diameter (Theo. II.). But the volume of the sphere is equal to its surface, or four great circles, multiplied by one third of its radius (Theo. X.), which is the same as one great circle multiplied by $\frac{4}{3}$ of its radius, or by $\frac{2}{3}$ of the diameter; hence, the volume of the sphere is equal to $\frac{2}{3}$ that of the circumscribed cylinder.

378. *Cor.* 1. Hence the sphere is to the circumscribed cylinder as 2 to 3; and their volumes are to each other as their surfaces.

379. *Cor.* 2. Since a cone is one third of a cylinder of the same base and altitude (Theo. V. Cor. 1), if a cone has the diameter of its base and its altitude each equal to the diameter of a given sphere, the volumes of the cone and sphere are to each other as 1 to 2; and the volumes of the cone, sphere, and circumscribing cylinder are to each other, respectively, as 1, 2, and 3.

EXERCISES

FOR ORIGINAL THOUGHT, ON REVIEW.

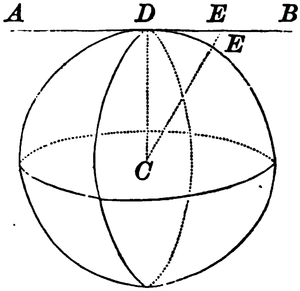
380. 1. The section of a cylinder made by a plane parallel to the bases is equal to either base.

2. A cylinder and a parallelopipedon of equal bases and altitude are equivalent to each other.

3. Two small circles equally distant from the center of a sphere are equal.

4. A plane perpendicular to a radius at its termination in the surface, is tangent to the sphere.

5. The side of either base of the largest rectangular prism that can be inscribed in a cylinder is to the half diameter of the base of the cylinder as the square root of 2 to 1.



6. The side of the largest cube that can be inscribed in a sphere is equal to the square root of one third of the square of the diameter of the sphere.

PRACTICAL APPLICATIONS.

MENSURATION.

DEFINITIONS.

381. Mensuration is the process of determining the measurement of lines, surfaces and volumes.

382. The **Measuring Unit** assumed for a given line some *linear unit*, as 1 inch, 1 foot; for a given surface, some *superficial unit*, as 1 square inch, 1 square foot; and for a given volume, some *cubic unit*, as 1 cubic inch, 1 cubic foot.

NOTE. The following **Problems,** at the discretion of the teacher, may be studied, after the completion of Book III. of the Geometry, in connection with the text to which they refer, as the learner progresses in the remainder of the work.

MENSURATION OF LINES AND SURFACES.

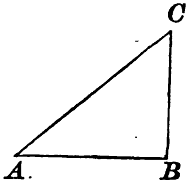
PROBLEM I. Any two sides of a RIGHT-ANGLED TRIANGLE being given, to find the third side.

To the square of the base add the square of the perpendicular; and the square root of the sum will give the hypothenuse (Theo. IX. Bk. IV.).

From the square of the hypothenuse subtract the square of the given side, and the square root of the difference will be the side required (Theo. IX. Cor. 1, Bk. IV.).

1. The base, $A\,B$, of the triangle $A\,B\,C$ is 48 feet, and the perpendicular, $B\,C$, 36 feet; what is the hypothenuse?

$48^2 + 36^2 = 3600$; $\sqrt{3600} = 60$ feet,
[the hypothenuse required.



2. The hypothenuse of a triangle is 53 feet, and the perpendicular 28 feet; what is the base?

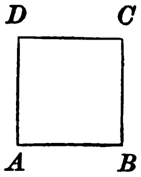
3. Two ships sail from the same port, one due west 50 miles, and the other due south 120 miles; how far are they apart? Ans. 130 miles.

4. A rectangular common is 25 rods long and 20 rods wide; what is the distance across it diagonally?

5. If a house is 40 feet long and 25 feet wide, with a pyramidal-shaped roof 10 feet in hight, how long is a rafter which reaches from the vertex of the roof to a corner of the building?

Problem II. To find the area of a PARALLELOGRAM.

Multiply the base by the altitude, and the product will be the area (Theo. V. Bk. IV.).



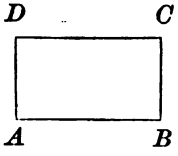
1. What is the area of a square, $ABCD$, whose side is 25 feet?

$25 \times 25 = 625$ feet, Ans.

2. What is the area of a square field whose side is 35.25 chains? Ans. 124 A. 1 R. 1 P.

3. How many square feet of boards are required to lay a floor 21 ft. 6 in. square?

4. Required the area of a square farm, whose side is 3,525 links.



5. What is the area of the rectangle $ABCD$, whose length, AB, is 56 feet, and whose width, AD, is 37 feet?

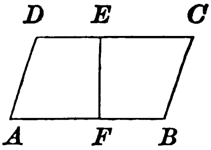
$56 \times 37 = 2{,}072$ sq. feet, Ans.

6. How many square feet in a plank, of a rectangular form, which is 18 feet long and 1 foot 6 inches wide?

7. How many acres in a rectangular garden, whose sides are 326 and 153 feet? Ans. 1 A. 23 P. $6\frac{1}{4}$ sq. yd.

8. A rectangular court 68 ft. 3 in. long, by 56 ft. 8 in. broad, is to be paved with stones of a rectangular form, each 2 ft. 3 in. by 10 in.; how many stones will be required?

Ans. $2{,}062\frac{2}{3}$ stones.



9. Required the area of the rhomboid *A B C D*, of which the side *A B* is 354 feet, and the perpendicular distance, *E F*, between *A B* and the opposite side *C D*, is 192 feet.

$354 \times 192 = 67,968$ feet, Ans.

10. How many square feet in a flower-plat, in the form of a rhombus, whose side is 12 feet, and the perpendicular distance between two opposite sides of which is 8 feet?

PROBLEM III. The area of a RECTANGLE and either of its sides being given, to find the other side.

Divide the area by the given side, and the quotient will be the other side.

If the rectangle be a square, the square root of the area will be the side.

1. The area of a rectangle is 2,072 sq. feet, and the length of one of the sides is 56 feet; what is the length of the other side?

$2072 \div 56 = 37$ feet, the side required.

2. How long must a rectangular board be, which is 15 inches in width, to contain 11 square feet?

3. A rectangular piece of land containing 6 acres is 120 rods long; what is its width? Ans. 8 rods.

4. What is the side of a square containing 625 square feet?

$\sqrt{625} = 25$ feet, the side required.

5. The area of a square farm is 124 A. 1 R. 1 P.; how many links in length is its side?

6. A certain corn-field in the form of a square contains 15 A. 2 R. 20 P. If the the corn is planted on the margin, 4 hills to a rod in length, how many hills are there on the margin of the field? Ans. 800 hills.

PROBLEM IV. The area of a RHOMBOID or RHOMBUS and the length of the base being given, to find the altitude; or the area and the altitude being given, to find the base.

Divide the area by the length of the base, and the quotient will be the altitude ; or

Divide the area by the altitude, and the quotient will be the base.

1. The area of a rhomboid is 67,968 square feet, and the length of the side taken as its base 354 feet; what is the altitude?

$67{,}968 \div 354 = 192$ feet, the altitude required.

2. The area of a piece of land in the form of a rhombus is 69,452 square feet, and the perpendicular distance between two of its opposite sides is 194 feet; required the length of one of the equal sides. Ans. 358 ft.

3. On a base 12 feet in length it is required to find the altitude of a rhomboid containing 968 square feet.

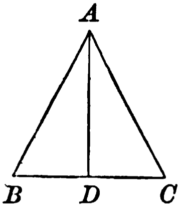
4. The area of a rhomboidal-shaped park is 1 A. 3 R. 34 P. 5½ yd.; and the perpendicular distance between the two shorter sides is 96 yards; required the length of each of these sides. Ans. 18 rods.

PROBLEM V. To find the area of a TRIANGLE.

If the base and altitude are given, multiply the base by half the altitude, or half the base by the altitude.

If the three sides are given, from half the sum of the three sides subtract each side; multiply the half sum and the three remainders together, and the square root of the product will be the area required.

1. Required the area of the triangle *A B C*, whose base, *B C*, is 210, and altitude, *A D*, is 190 feet.



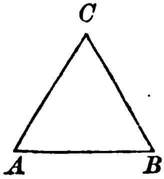
$210 \times \frac{190}{2} = 19{,}950$ square feet, the area
[required.

2. A piece of land is in the form of a right-angled triangle, having the sides about the right angle, the one 254 and the other 136 yards; required the area in acres. Ans. 3 A. 2 R. 10 P. 29½ yd.

3. Required the number of square feet in a triangular board whose base is 27 inches, and altitude 27 feet.

4. What is the area of a triangle whose base is 15.75 chains, and the altitude 10.22 chains?

5. What is the area of a triangle, ABC, whose sides, AB, BC, CA, are 40, 30, and 50 feet?



$\overline{30 + 40 + 50} \div 2 = 60$, half the sum of
[the three sides.

$60 - 30 = 30$, first remainder.
$60 - 40 = 20$, second remainder.
$60 - 50 = 10$, third remainder.

$60 \times 30 \times 20 \times 10 = 360{,}000$; $\sqrt{360{,}000} = 600$, square feet, the area required.

6. How many square feet in a triangular floor, whose sides are 15, 16, and 21 feet?

7. Required the area of a triangular field whose sides are 834, 658, and 423 links. Ans. 1 A. 1 R. 20 P. 4 yd. 1.6 ft.

8. Required the area of an equilateral triangle, of which each side is 15 yards.

9. What is the area of a garden in the form of a parallelogram, whose sides are 432 and 263 feet, and a diagonal 342 feet? Ans. 2 A. 10 P. 11.46 yd.

PROBLEM VI. The area and the base of a TRIANGLE being given, to find the altitude; or the area and altitude being given, to find the base.

Divide double the area by the base, and the quotient will be the altitude; or divide double the area by the altitude, and the quotient will be the base.

1. The area of a triangle is 1300 square feet, and the base 65 feet; what is the altitude?

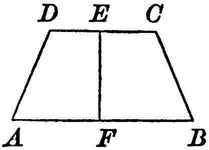
$1300 \times 2 = 2600$; $2600 \div 65 = 40$ ft., altitude required.

2. The area of a right-angled triangle is 17,272 yards, of which one of the sides about the right angle is 136 yards; required the other perpendicular side.

3. The area of a triangle is 46.25 chains, and the altitude 5.2 chains; what is the base?

Problem VII. To find the area of a TRAPEZOID.

Multiply half the sum of its parallel sides by its altitude (Theo. VII. Bk. IV.).



1. What is the area of the trapezoid $ABCD$, whose parallel sides, AB, DC, are 32 and 24 feet, and the altitude, EF, 20 feet?

$32 + 24 = 56$; $56 \div 2 = 28$; $28 \times$ [$20 = 560$ sq. ft., the area required.

2. How many square feet in a board in the form of a trapezoid, whose width at one end is 2 feet 3 inches, and at the other 1 foot 6 inches, the length being 16 feet?

3. Required the area of a garden in the form of a trapezoid, whose parallel sides are 786 and 473 links, and the perpendicular distance between them 986 links.

Ans. 6 A. 33 P. 3 sq. yd.

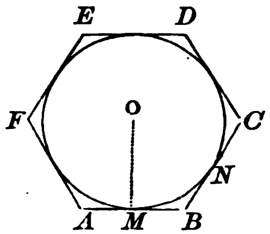
Problem VIII. To find the area of a REGULAR POLYGON.

If the perimeter and apothegm are given, multiply the perimeter by half the apothegm, and the product will be the area (Theo. XXV. Bk. IV.).

If the perimeter and side are given, multiply the square of the side of the polygon by the area of a similar polygon whose side is unity or 1 (Theo. XXI. Bk. IV.).

383. A Table of Regular Polygons whose Side is 1.

NAMES.	AREAS.	NAMES.	AREAS.
Triangle,	0.4330127	Octagon,	4.8284271
Square,	1.0000000	Nonagon,	6.1818242
Pentagon,	1.7204774	Decagon,	7.6942088
Hexagon,	2.5980762	Undecagon,	9.3656399
Heptagon,	3.6339124	Dodecagon,	11.1961524



1. Required the area of a regular hexagon, *A B C D E F*, whose sides, *A B*, *B C*, &c. are each 15 yards, and the apothegm, *O M*, 13 yards.
$15 \times 6 = 90$; $90 \times \frac{13}{2} = 585$ yd.,
[the area required.

2. What is the area of a regular pentagon, whose sides are each 25 feet, and the perpendicular from the center to a side 17,205 feet?

3. A park is laid out in the form of a regular heptagon, whose sides are each 19.263 chains; and the perpendicular distance from the center to each of the sides is 20 chains. How many acres does it contain? Ans. 134 A. 3 R. 14 P.

4. Required the area of an equilateral triangle, whose side is 100 feet.

$100^2 = 10,000$; $10,000 \times 0.4330127 = 4330.127$ square
[feet, the area required.

5. What is the area of a regular pentagon, whose side is 37 yards?

6. How many acres in a field in the form of a regular undecagon, whose side is 27 yards?

Ans. 1 A. 1 R. 25 P. 21 sq. yd. 2.7 sq. ft.

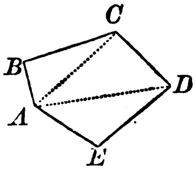
PROBLEM IX. To find the area of an IRREGULAR POLYGON.

Divide the polygon into triangles, or triangles and trapezoids, and find the areas of each of them separately; the sum of these areas will be the area required.

If the irregular polygon is a quadrilateral, the area may be found by multiplying together the diagonal and half the sum of the perpendiculars drawn from it to the opposite angles.

1. Required the area of the irregular pentagon *A B C D E*, of which the diagonal *A C* is 20 feet, and *A D* 36 feet; and the perpendicular distance from the angle *B* to *A C* is 8 feet,

from C to AD 12 feet, and from E to AD 6 feet.



$20 \times \frac{8}{2} = 80$; $36 \times \frac{12}{2} = 216$; $36 \times \frac{6}{2} = 108$; $80 + 216 + 108 = 404$ sq. ft., the area required.

2. What is the area of a trapezium, whose diagonal is 42 feet, and the two perpendiculars from the diagonal to the opposite angles are 16 and 18 feet?

3. In an irregular hexagon, $ABCDEF$, are given the sides AB 536, BC 498, CD 620, DE 580, EF 398, and AF 492 links, and the diagonals AC 918, CE 1048, and AE 652 links; required the area.

Ans. 6 A. 2 R. 9 P. 23 sq. yd. 8.4 sq. ft.

4. In measuring along one side, AB, of a quadrangular field, $ABCD$, that side and the perpendiculars let fall on it from two opposite corners measured as follows: AB 1110, AE 110, AF 745, DE 352, CF 595 links. What is the area of the field? Ans. 4 A. 1 R. 5 P. 24 sq. yd.

PROBLEM XI. To find the area of a CIRCLE.

If the circumference and radius are given, multiply the circumference by half the radius (Theo. XXXII. Bk. IV.).

If the diameter or radius be given, multiply the square of the half diameter or radius by 3.1416 (Theo. XXXII. Cor. 2. Bk. IV.).

Scholium. Multiplying the circumference by half the radius is the same as multiplying the circumference and diameter together, and taking one fourth of the product. Hence the area of the circle may also be found by *multiplying the square of the diameter by* 0.7854; *or by multiplying the square of the circumference by* 0.07958.

1. The circumference of a circle is 314.16 feet, and its radius 50 feet; what is its area?

$314.16 \times \frac{50}{2} = 7854$ feet, the area required.

2. If the circumference of a circle is 355 feet, and its diameter 113 feet, what is the area?

3. What is the area of a circular garden, whose radius is 281½ links? Ans. 2 A. 1 R. 38 rd. 9 yd. 5 ft.

4. A horse is tethered in a meadow by a cord 39.25075 yards long; over how much ground can he graze?

5. Required the area of a semicircle, the diameter of the whole circle being 751 feet. Ans. 5 A. 13 P. 16 sq. yd.

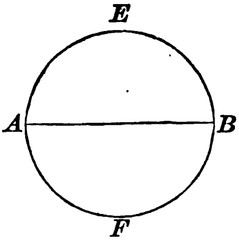
Problem X. To find the diameter or circumference of a circle.

If the circumference is given, divide it by 3.1416, *and the quotient will be the diameter* (Theo. XXXII. Cor. 3, Bk. IV.).

If the diameter is given, multiply it by 3.1416, *and the product will be the circumference.*

If the area is given, divide the area by 0.7854, *and the square root of the quotient will be the diameter; or, divide the area by* 0.07958, *and the square root of the quotient will be the circumference.*

The diameter may also be found by multiplying the circumference by .31831, the reciprocal of 3.1416.



1. The diameter, AB, of the circle $AEBF$ is 100 feet; what is its circumference?

$100 \times 3.1416 = 314.16$ feet, the [cumference required.

2. Required the circumference of a circle whose diameter is 628 links.

Ans. 1 fur. 38 rd. 5 yd. 1.56 in.

3. If the diameter of the earth is 7912 miles, what is its circumference?

4. Required the diameter of a circular pond whose circumference is 928 rods. Ans. 7 fur. 15 rd. 2 yd. 5.55 in.

5. The area of a circle is 314.16 feet; what is its diameter?

$314.16 \div 0.7854 = 400$; $\sqrt{400} = 20$ feet, the diameter re[quired.

6. What must be the length of a cord to be used as a radius in describing a circle which shall contain exactly 1 acre?

7. The area of a circular pond is 6 A. 1 R. 27 P. 18.2 yd.; what is the circumference? Ans. 625 yd.

8. The area of a circle is 7856 feet; what is the circumference?

9. The length of a rectangular garden is 32, and its width 18 rods; required the diameter of a circular garden having the same area. Ans. 27 rd. 1 sq. ft. 4 in.

PROBLEM XII. The diameter or circumference of a CIRCLE being given, to find the side of an EQUIVALENT SQUARE.

Multiply the diameter by 0.8862, *or the circumference by* 0.2821; *the product in either case will be the side of an equivalent square.*

For, since 0.7854 is the area of a circle whose diameter is 1 (Prob. X. Sch.), the square root of 0.7854, which is 0.8862, is the side of a square which is equivalent to a circle whose diameter is 1. Now when the circumference is 1, the side of an equivalent square must have the same ratio to 0.8862 as the diameter 1 has to its circumference 3.1416 (Theo. XXXII. Cor. 4, Bk. VI.); and 0.8862 ÷ 3.1416 gives 0.2821 as the side of the equivalent square when the circumference is 1.

1. The diameter of a circle is 120 feet; what is the side of an equivalent square?

$$120 \times 0.8862 = 106.344 \text{ feet, the side required.}$$

2. The circumference of a circle is 100 yards; what is the side of an equivalent square? Ans. 28.21 yd.

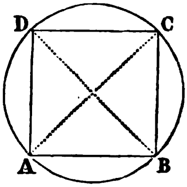
3. There is a circular floor 30 feet in diameter; what is the side of a square floor containing the same area?

4. If 500 feet is the circumference of a circular island, what is the side of a square of equal area? Ans. 141.05 ft.

PROBLEM XIII. The diameter or circumference of a CIRCLE being given, to find the side of the INSCRIBED SQUARE.

Multiply the diameter by 0.7071, *or the circumference* *by* 0.2251; *the product in either case will be the side of the inscribed square.*

For, 0.7071 is the side of the inscribed square when the diameter of the circumscribed circle is 1, since the side of the inscribed square is to the radius of the circle as the square root of 2 to 1 (Theo. XXIII. Cor. 2, Bk. IV.); consequently, the side is to the diameter, or twice the radius, as half the square root of 2 is to 1, and half the square root of 2 is 0.7071, approximately. Now, the ratio of the diameter of a circle to the side of its inscribed square being as 1 to 0.7071, and the ratio of the circumference of a circle to its diameter as 3.1416 to 1, the ratio of the inscribed square is to the circumference of the circle as 0.7071 to 3.1416; and 0.7071 ÷ 3.1416 gives 0.2251 as the side of the inscribed square when the circumference is 1.



1. The diameter, *A C*, of a circle is 110 feet; what is the side, *A B*, of the inscribed square?

$110 \times 0.7071 = 77.781$ feet, the side required.

2. The circumference of a circle is 300 feet; what is the side of the inscribed square?

Ans. 67.53 ft.

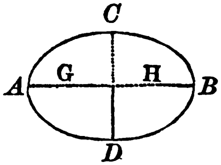
3. A log is 36 inches in diameter; of how many inches square can a stick be hewn from it?

4. There is a circular field 1000 rods in circuit; what is the side of the largest square that can be described in it?

Ans. 225.10 rods.

The Ellipse.

384. An Ellipse is a plane figure bounded by a curve, from any point of which the sum of the distances to two fixed points is equal to a straight line drawn through those two points, and terminated both ways by the curve.



Thus *A D B C* is an ellipse. The two

fixed points G and H are called the *foci*. The longest diameter, $A\,B$, of the ellipse is called its *major* or *transverse axis*, and its shortest diameter, $C\,D$, is called its *minor* or *conjugate axis*.

385. The Area of an ellipse is a mean proportional between the areas of two circles whose diameters are the two axes of the ellipse.

PROBLEM XIV. To find the area of an ELLIPSE, the major and minor axes being given.

Multiply the axes together, and their product by 0.7854, *and the result will be the area.*

For $A\,B^2 \times 0.7854$ expresses the area of a circle whose diameter is $A\,B$, and $C\,D^2 \times 0.7854$ expresses the area of a circle whose diameter is $C\,D$; and the product of these two areas is equal to $A\,B^2 \times C\,D^2 \times 0.7854^2$, which is equal to the square of $A\,B \times C\,D \times 0.7854$; hence, $A\,B \times C\,D \times 0.7854$ is a mean proportional between the areas of the two circles whose diameters are $A\,B$ and $C\,D$ (Theo. III. Bk. II.); consequently it measures the area of an ellipse whose axes are $A\,B$ and $C\,D$ (Art. 429).

1. Required the area of an ellipse, of which the major axis is 60 feet, and the minor axis 40 feet.

$60 \times 40 \times 0.7854 = 1884.96$ sq. ft., the area required.

2. What is the area of an ellipse whose axes are 75 and 35 feet?

3. Required the area of an ellipse whose axes are 526 and 354 inches. Ans. 112 yd. 7 ft. 84.62 in.

4. How many acres in an elliptical pond whose semi-axes are 436 and 254 feet?

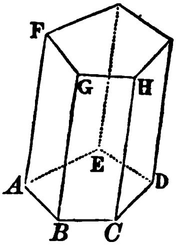
Ans. 7 A. 3 R. 37 P. 27 sq. yd. 7 sq. ft.

MEASUREMENT OF VOLUMES.

PROBLEM XV. To find the surface of a RIGHT PRISM.

Multiply the perimeter of the base by the altitude, and the product will be the CONVEX *surface* (Theo. I. Bk. VI.).

To this add the areas of the two bases, and the result will be the ENTIRE *surface.*



1. Required the convex surface of a pentangular prism, having each side of its base, $ABCDE$, equal to 2 feet, and its altitude, AF, equal to 5 feet.
$2 \times 5 = 10$; $10 \times 5 = 50$ square feet, the [convex surface required.

2. The altitude of a hexangular prism is 12 feet, two of its faces are each 2 feet wide, three are each $2\frac{1}{2}$ feet wide, and the remaining face is 9 inches wide; what is the convex surface of the prism?

3. Required the entire surface of a cube, the length of each edge being 25 feet.

4. Required, in square yards, the wall surface of a rectangular room, whose hight is 20 feet, width 30 feet, and length 50 feet. Ans. $355\frac{5}{9}$ sq. yd.

PROBLEM XVI. To find the volume of a PRISM.

Multiply the area of its base by its altitude, and the product will be its volume (Theo. IX. Bk. VI.).

1. Required the volume of a pentangular prism, having each side of its base equal to 2 feet, and its altitude equal to 5 feet.
$2^2 \times 1.72048 = 6.88192$; $6.88192 \times 5 = 34.40960$ cubic feet, [the volume required.

2. Required the volume of a triangular prism, whose length is 10 feet, and the three sides of whose base are 3, 4, and 5 feet. Ans. 60.

3. A slab of marble is 8 feet long, 3 feet wide, and 6 inches thick; required its volume.

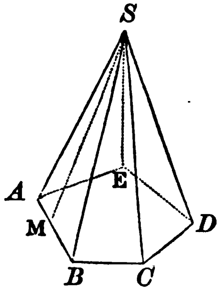
4. There is a cistern in the form of a cube, whose edge is 10 feet; what is its capacity in liquid gallons?
Ans. 7480.519 gallons.

5. Required the volume of a quadrilateral prism, the length being 19 feet, the sides of the base 43, 54, 62, and 38, and the diagonal between the first and second sides, 70 inches.
Ans. 306.047 cu. ft.

PROBLEM XVII. To find the surface of a RIGHT PYRAMID.

Multiply the perimeter of the base by half its slant hight, and the product will be the CONVEX *surface* (Theo. XVII. Bk. V.). *To this add the area of the base, and the result will be the* ENTIRE *surface.*

Scholium. The surface of an oblique pyramid is found by taking the sum of the areas of its several faces.



1. Required the convex surface of a pentangular pyramid, *A B C D E – S*, each side of whose base, *A B C D E*, is 5 feet, and whose slant hight, *S M*, is 20 feet.

$5 \times 5 = 25$; $25 \times \frac{20}{2} = 250$ square feet,
[the surface required.

2. What is the entire surface of a triangular pyramid, of which the slant hight is 18 feet, and each side of the base 42 inches? Ans. 99.804 sq. ft.

3. Required the convex surface of a triangular pyramid, the slant hight being 20 feet, and each side of the base 3 feet.

4. What is the entire surface of a quadrangular pyramid, the sides of the base being 40 and 30 inches, and the slant hight upon the greater side 20.04, and upon the less side 20.07 feet? Ans. 125.308 ft.

PROBLEM XVIII. To find the surface of a FRUSTUM OF A RIGHT PYRAMID.

Multiply half the sum of the perimeters of its two bases by its slant hight, and the product will be the CONVEX *surface* (Theo. XIX. Bk. V.); *to this add the areas of the two bases, and the result will be the* ENTIRE *surface.*

1. What is the entire surface of a rectangular frustum whose slant hight is 12 feet, and the sides of whose bases are 5 and 2 feet?

$5 \times 4 = 20$; $2 \times 4 = 8$; $20 + 8 = 28$; $^{28} \times 12 = 168$;
$5^2 + 2^2 = 29$; $168 + 29 = 197$ sq. ft., area required.

2. Required the convex surface of a regular hexangular frustum, whose slant hight is 16 feet, and the sides of whose bases are 2 feet 8 inches and 3 feet 4 inches.

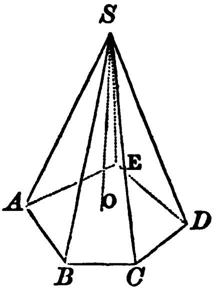
3. What is the entire surface of a regular pentangular frustum, whose slant hight is 11 feet, and the sides of whose bases are 18 and 34 inches? Ans. 136.849 sq. ft.

Problem XIX. To find the volume of a PYRAMID.

Multiply the area of its base by one third of its altitude (Theo. XXII. Bk. V.).

1. Required the volume of a pentangular pyramid, $ABCDE-S$, each side of whose base, $ABCDE$, is 5 feet, and whose altitude, SO, is 15 feet.

$5^2 \times 1.7205 = 43.0125$; $43.0125 \times \frac{15}{3} = 215.0575$ cu. ft., the volume required.



2. What is the volume of a hexangular pyramid, the altitude of which is 9 feet, and each side of the base 29 inches?

3. What is the volume of a square pyramid, each side of whose base is 30 feet, and whose perpendicular hight is 25 feet? Ans. 7500.

4. Required the volume of a triangular pyramid, the perpendicular hight of which is 24 feet, and the sides of the base 34, 42, and 50 inches. Ans. 39.2354 cu. ft.

Problem XX. To find the volume of a FRUSTUM OF A PYRAMID.

Add together the areas of the two bases and a mean proportional between them, and multiply that sum by one third of the altitude of the frustum (Theo. XXIII. Bk. V.).

1. Required the volume of the frustum of a quadrangular pyramid, the sides of whose bases are 3 feet and two feet, and whose altitude is 15 feet.

$3 \times 3 = 9$; $2 \times 2 = 4$; $\sqrt{9 \times 4} = 6$ (Theo. III. Bk. II.); $(9 + 4 + 6) \times \frac{15}{3} = 95$ cu. ft., volume required.

2. How many cubic feet in a stick of timber in the form of a quadrangular frustum, the sides of whose bases are 15 inches and 6 inches, and whose altitude is 20 feet?

3. Required the volume of a pentangular frustum, whose altitude is 5 feet, each side of whose lower base is 18 inches, and each side of whose upper base is 6 inches. Ans. 9.319 cu. ft.

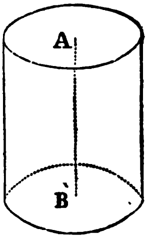
4. Required the volume of the frustum of a triangular pyramid, the altitude of which is 14 feet, the sides of the lower base 21, 15, and 12, and those of the upper base 14, 10, and 8 feet. Ans. 868.752 cu. ft.

Problem XXI. To find the surface of a cylinder.

Multiply the circumference of its base by its altitude, and the product will be the CONVEX *surface* (Theo. I. Bk. VI.). *To this add the areas of its two bases, and the result will be the* ENTIRE *surface.*

1. What is the entire surface of a cylinder, the altitude of which, *A B*, is 10 feet, and the circumference of the base 20 feet?

$10 \times 20 = 200$; $20^2 \times 0.07958 \times 2 = 63.264$; $200 + 63.264 = 263.264$ sq. ft., the surface required.



2. Required the convex surface of a cylinder whose altitude is 16 feet, and the circumference of whose base is 21 feet.

3. What is the entire surface of a cylinder whose altitude is 10 inches, and whose circumference is 4 feet?

4. How many times must a cylinder 5 feet 3 inches long, and 21 inches in diameter, revolve, to roll an acre?

Ans. 1509.18 times.

Problem XXII. To find the volume of a cylinder.

Multiply the area of the base by the altitude, and the product will be the volume (Theo. II. Bk. VI.).

1. What is the volume of a cylinder, whose altitude is 10 feet. and the circumference of whose base is 20 feet?

$20^2 \times 0.07958 \times 10 = 318.32$ cu. ft., volume required.

2. Required the volume of a cylindrical log, whose length is 9 feet, and the circumference of whose base is 6 feet.

Ans. 25.7831 cu. ft.

3. The Winchester bushel is a hollow cylinder $18\frac{1}{2}$ inches in diameter, and 8 inches deep; what is its capacity in cubic inches?

PROBLEM XXIII. To find the surface of a CONE.

Multiply the circumference of the base by half the slant hight (Theo. III. Bk. VI.), *and the product will be the convex surface. To this add the area of the base, and the result will be the entire surface.*

1. What is the convex surface of a cone, whose slant hight is 28 feet, and the circumference of whose base is 40 feet?

$40 \times \frac{28}{2} = 560$ sq. ft., the surface required.

2. Required the entire surface of a cone, whose slant hight is 14 feet, and the circumference of whose base is 92 inches.

3. What is the surface of a cone, whose slant hight is 9 feet, and the diameter of whose base is 36 inches?

4. How many yards of canvas are required for the covering of a conical tent, the slant hight of which is 30 feet, and the circumference of the base 900 feet? Ans. 1500 sq. yd.

PROBLEM XXIV. To find the surface of a FRUSTUM OF A CONE.

Multiply half the sum of the circumferences of its two bases by its slant hight, and the product will be the convex surface (Theo. IV. Bk. VI.). *To this add the area of its bases, and the result will be the entire surface.*

Scholium. The convex surface of a frustum of a cone may also be found by *multiplying the slant hight* by the circumference of a section at equal distances between the two bases (Theo. IV. Cor., Bk. VI.).

1. Required the convex surface of a frustum of a cone, whose slant hight is 20 feet, and the circumferences of whose bases are 30 feet and 40 feet.

$\frac{30+40}{2} \times 20 = 700$ sq. ft., the surface required.

2. Required the surface of a frustum of a cone, the diameters of the bases being 43 inches and 23 inches, and the slant hight 9 feet.

3. What is the convex surface of a frustum of a cone, of which a section equidistant from its two bases is 24 feet in circumference, the slant hight of the frustum being 19 feet?

4. From a cone the circumference of whose base is 10 feet, and whose slant hight is 30 feet, a cone has been cut off, whose slant hight is 8 feet. What is the convex surface of the frustum? Ans. $139\frac{1}{3}$ sq. ft.

PROBLEM XXV. To find the volume of a CONE.

Multiply the area of its base by one third of its altitude, and the product will be the volume (Theo. V. Bk. VI.)

1. What is the volume of a cone whose altitude is 42 feet, and the diameter of whose base is 10 feet?

$10^2 \times 0.7854 \times \frac{42}{3} = 1099.56$ cu. ft., volume required.

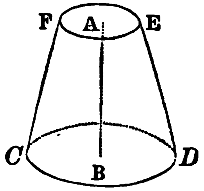
2. Required the volume of a cone whose altitude is 63 feet, and the radius of whose base is 12 feet 6 inches.

3. How many cubic feet in a conical stick of timber, whose length is 18 feet, the diameter at the larger end being 42 inches? Ans. 57.7269 cu. ft.

PROBLEM XXVI. To find the volume of the FRUSTUM OF A CONE.

Add together the areas of the two bases and a mean proportional between them, and multiply that sum by one third of the altitude of the frustum; and the result will be the volume required (Theo. VI. Bk. VI.).

1. What is the volume of a frustum of a cone, *C D E F*, whose altitude, *A B*, is 21 feet, and the area of whose bases, *F E*, *C D*, are 80 square feet and 300 square feet?



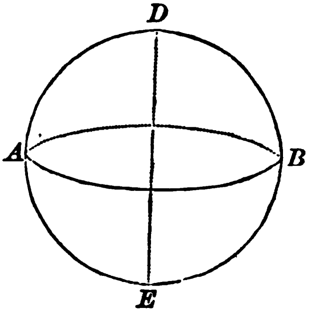
$(80 + 300 + \sqrt{80 \times 300}) \times \frac{21}{3} =$ 3732.96 cu. ft., volume required.

2. Required the volume of a frustum of a cone, the diameters of the bases being 38 and 27 inches, and the altitude 11 feet.

3. If a cask, which is two equal frustums of cones joined together at the larger bases, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches, how many gallons of wine will it hold? Ans. 79.06.

PROBLEM XXVII. To find the surface of a SPHERE.

Multiply the diameter by the circumference of a great circle of the sphere (Theo. IX. Bk. VI.); *or multiply the area of one great circle of the sphere by* 4 (Theo. VIII. Cor. 1, Bk. VI.); *or multiply* 3.1416 *by the square of the diameter* (Theo. IX. Cor. 4, Bk. VI.).



1. What is the surface of a sphere, whose diameter, $E\,D$, is 40 feet, and whose circumference, $A\,E\,B\,D$, is 125.664?
125.664 × 40 = 5026.56 sq. ft., [the surface required.

2. Required the surface of a sphere whose diameter is 30 inches.

3. What is the surface of a globe whose diameter is 7 feet and circumference 21.99 feet? Ans. 153.93.

4. How many square miles of surface has the earth, its diameter being 7912 miles?

PROBLEM XXVIII. To find the volume of a SPHERE.

Multiply the surface of the sphere by one third of its radius (Theo. X. Bk. VI.); *or multiply the cube of the diameter of the sphere by* 0.5236 (Theo. X. Cor. 5, Bk. VI.).

1. What is the volume of a sphere whose diameter is 40 inches?

$40^3 \times 0.5236 = 33510.4$ cu. in., the volume required.

2. Required the solidity of a globe whose circumference is 60 inches.

3. What is the solidity of the moon in cubic miles, supposing it a perfect sphere with a diameter of 2160 miles?

4. Required the solidity of the earth, supposing it to be a perfect sphere, whose diameter is 7912 miles.

Ans. 259332805349.80493 cu. miles.

MISCELLANEOUS EXERCISES.

386. 1. There is a park in the form of a square containing 10 acres; how many rods less is the distance from the center to each corner, than the length of the side of the square?

Ans. 11.716 rods.

2. How many many acres in a rhomboidal field, of which the sides are 1234 and 762 links, and the perpendicular distance between the longer sides of which is 658 links?

Ans. 8 A. 19 P. 4 sq. yd. 6¼ sq. ft.

3. Required the area of an isosceles triangle, whose base is 25 and each of its equal sides 40 rods.

4. What is the area of a rhomboidal field, whose sides are 57 and 83 rods, and the diagonal 127 rods?

Ans. 22 A. 3 R. 21 P. 26 yd. 5 ft.

5. The area of a rectangular farm is 266 A. 3 R. 8 P., and the breadth 46 chains; what is the length? Ans. 58 chains.

6. A triangular field contains 30 A. 3 R. 27 P.; one of its sides is 97 rods; required the perpendicular distance from the opposite angle to that side. Ans. 102 rods.

7. What is the area of a triangular field whose base is 97 rods, and the perpendicular distance from the base to the opposite angle 40 rods? Ans. 12 A. 20 P.

8. What is the side of a square containing 625 square feet?

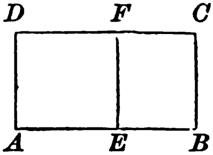
Ans. 25 feet.

9. Required the area of a garden in the form of a trapezoid,

whose parallel sides are 786 and 473 links, and the perpendicular distance between them 986 links.

Ans. 6 A. 33 P. 3 yd.

10. How many acres in a quadrilateral field, having two parallel sides 83 and 101 rods in length, and which are distant from each other 60 rods?



11. If the sides of a rectangle, *ABCD*, are 25 and 14 feet, how wide an area, *EBCF*, to contain 154 square feet, can be cut off by a line parallel to the side *AD*? Ans. 11 feet.

12. A farmer has a field 16 rods square, and wishes to cut off from one side a rectangular lot containing exactly one acre; what must be the width of the lot?

13. A carpenter sawed off, from the end of a rectangular plank, in a line parallel to its width, 5 square feet. From the remainder he then sawed off, in a line parallel to the length, 8 square feet. Required the dimensions of the part still remaining, provided the original dimensions of the plank were 20 feet by 15 inches. Ans. 16 feet by 9 inches.

14. In a four-sided rectilineal field, *A B C D*, on account of obstructions, there could be taken only the following measures: the two sides *B C* 265 and *A D* 220 yards, the diagonal *A C* 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, *A E* 100, and *C F* 70 yards. Required the area in acres.

15. What is the volume of a regular triangular pyramid whose edge is 24 yards? Ans. 1629.158 cu. yd.

16. Required the volume of the largest rectangular beam that can be sawed from a cylindrical log 20 feet long and 14 inches in diameter.

17. Required the capacity of a tub in the form of a frustum of a cone, of which the greatest diameter is 48 inches, the inside length of the staves 30 inches, and the diagonal between the farthest extremities of the diameters 50 inches.

Ans. 165.34 gals.

"STANDARD AND ENDURING."

GREENLEAF'S

New Mathematical Series.

MERITORIOUS AND ECONOMICAL.

Tested and proved, these books have an unprecedented popularity.

Over 1,600,000 Copies in use!

DISTINGUISHING FEATURES.

1. A complete course for all classes of learners, upon the same plan and by one author.
2. The Primary Books are strictly elementary, and not mere abstracts or abridgments of larger works.
3. Each book is complete in itself, but so constructed as to prepare the learner for systematic progress.
4. The methods and processes are entirely modern, and most happily combine science and practice.
5. The course is arranged with a view of saving the time of the learner, and of expense, by the use of the smallest number of text-books.
6. The several books are beautifully illustrated, printed upon sized paper, and handsomely and durably bound.

SCHOOL OFFICERS

Contemplating a change of Mathematical Text-Books, are invited to examine GREENLEAF'S NEW SERIES. Address,

ROBERT S. DAVIS & CO.,

36 Bromfield St., Boston.

GREENLEAF'S

NEW SERIES OF ARITHMETICS.

Each Book is complete in itself, and sold separately.

Superior to all others in use.

1. In clearness of statement.
2. In progressive development.
3. In logical arrangement.
4. In conciseness and accuracy.
5. In simplicity of analysis.
6. In improved processes.
7. In conformity to business usages.
8. In variety of practical problems.
9. In conformity to new legal standards.
10. In adaptation to all classes of pupils.

NEW PRIMARY ARITHMETIC.

This book is on the plan of Object Teaching, and is beautifully illustrated.

It combines Written and Oral Exercises, in lessons, all of an inviting and attractive nature.

Primary Classes are fully prepared by its use for the next book of the series.

NEW ELEMENTARY ARITHMETIC.

A Comprehensive Manual, at once inductive, analytic and useful, expressly prepared to provide for new educational demands.

It is especially adapted to Intermediate Classes, and to the wants of pupils whose limited opportunities require that they should learn much in the shortest possible time.

It brings together in a brief form the essentials of arithmetic, mental and written, the fundamental rules, fractions, percentage and interest, and with the New Primary Arithmetic forms a complete course, in only two books.

NEW PRACTICAL ARITHMETIC.

A COMPLETE ARITHMETIC, entirely modern in its methods, and combining throughout practical utility with scientific accuracy.

It was the first to make written exercises intellectual, to thoroughly enforce educational results by means of systematic review questions and exercises, and to introduce certain TECHNICAL WORK of great value to Business Men, Farmers, Masons and Carpenters.

The extent to which this book has caused like books to be revised or re-written, is a sufficient evidence of its standard character.

Its important features, such as the early introduction of the decimal point, the avoiding of many special rules, and of obsolete or extraneous matter, and the introduction of valuable new applications, have aided to make this work the most popular of its kind now before the public.

In it may be found, treated in a practical manner, the Metric System, Stocks and Government Securities, new par of English Exchange, and the new legal values of Foreign Coins.

It will be noted that this work, with Greenleaf's New Primary and New Elementary Arithmetics, forms a clear, practical, comprehensive system, a time-saving and labor-saving course,

COMPLETE IN ONLY THREE BOOKS,

and is believed to be the cheapest, most convenient and productive, that has ever appeared.

By their use, in comparison with some *extended* courses of five or more books, it is safe to say that the learner may not only make better progress, but at least

Save Two Years of Valuable Time!

GREENLEAF'S ARITHMETICS

have been approved for, and are now used, in the most important Schools of

BOSTON, NEW YORK CITY, PHILADELPHIA,

and many other cities; and have been recently introduced into upwards of **6,800 Schools** in the EASTERN AND MIDDLE STATES, including *entire uniformity* in numerous counties.

GREENLEAF'S is the standard in upwards of **1,000** CITIES and TOWNS in the NEW ENGLAND STATES alone.

TESTIMONIALS.

From REV. LEVERETT GRIGGS, *School Visitor, Bristol, Conn.*, April 3, 1875.—"Some years since, Greenleaf's Arithmetics were used in our schools. They were displaced by ————. Last Fall the Board unanimously adopted Greenleaf's New Series, to the great satisfaction of our most accomplished teachers, as well as that of our school officers."

From REV. A. E. DENNISON, *Acting School Visitor, Plainville, Conn.*, June, 1875.—"Greenleaf's New Series of Arithmetics give satisfaction. We are well satisfied with the change from ————."

From S. P. WILLIAMS, A. M., *Principal of Union Graded School, Plainville, Conn.*, June, 1875.—"Greenleaf's New Series has many merits. I am much pleased with the change."

From NATHANIEL WILCOXEN, *Chairman of School Board, Oxford, Conn.*, March 11, 1875.—"I have given Greenleaf's New Series of Mathematics a careful examination. I find it, I may well say, the best adapted to our schools of any with which I am acquainted."

From LINUS T. GILBERT, *School Visitor, Litchfield, Conn.*, Jan. 2, 1875.—"Greenleaf's New Series of Arithmetics are just what we have wanted for some time. I think them *far superior* to ————, formerly in use. I am pleased that they have been introduced in our schools. There are many new features in the books that must commend them to towns now using books like ————, as we formerly had."

GREENLEAF'S NEW MATHEMATICAL SERIES.

From S. T. DUTTON, *Principal South Norwalk School, Ct.*, April, 1875.—"I have used Greenleaf's Mathematical Works, and take pleasure in testifying to the rare excellencies which they possess. So well are they adapted to the popular wants that I never heard a teacher speak of them as deficient in any respect."

From E. T. HAYWARD, *Principal W. District Grammar School, Rockville, Ct.*, April, 1875. "Having tested Greenleaf's New Arithmetics in my school for a term of fourteen weeks, I cheerfully recommend them to the favorable consideration of all teachers in need of good text-books. The prominence given to the principles embodied in the fundamental rules, and the constant attention to practical applications throughout the works are prominent excellencies."

From HEMAN R. TIMLOW, *Acting School Visitor, Southington, Ct.*, Dec., 1874.—"I have examined with care Greenleaf's New Arithmetics, and for three months they have been in use in the eleven school districts of this town. The period of use is too brief to justify the commendation a longer experience might contribute. But after repeated personal examinations of the series, and quietly watching the fruits as they appear, I am prepared to defend the change the School Board made last spring; and not only this, but I feel warranted in stating the conviction that the Greenleaf's New Series is surpassed by none in the country"

From W. E. HAWKINS, *East Hartford, Conn.*, April, 1875.—"I have used Greenleaf's Arithmetics for some years, and can heartily recommend the New Series as meeting the wants of the school room in every particular."

From L. N. CUMMINGS, *Principal of High School, Portland, Conn.*, April 29, 1874.—"I have recently introduced into my school Greenleaf's New Practical Arithmetic. Having previously examined many other editions, I can truly say it is the most instructive and complete of any Arithmetic I have ever seen, and I would heartily recommend it to the consideration of teachers and school officers everywhere."

From R. B. CLARKE, *Fitchburg*, Jan. 23, 1875.—"We have used Greenleaf's Series, as a whole, or in part, for several years, with entire satisfaction. In my opinion they have not been surpassed in excellence as text-books."

From FRANK M. WILKINS, *Principal of High School, Canton, Mass.*—"We use in our school Greenleaf's National Arithmetic, Algebra, Geometry and Trigonometry. I know of no series of text-books on any subject which I use with so much pleasure and satisfaction in school. The Algebra especially is above praise."

GREENLEAF'S NEW MATHEMATICAL SERIES.

From Percy S. Bryant, *Principal of High School, Thompsonville, Conn.*—"I have carefully examined Greenleaf's New Mathematical Series, and consider it superior to any with which I am familiar. I gladly recommend it to all who desire first-class text-books."

From H. P. Topliff, *Principal, Graded School, Brooklyn, Conn.*—"I consider Greenleaf's New Elementary Algebra the *best* of the best, having used it as a text-book in the school-room for some years. It is the text-book on Algebra adopted for this town."

From A. L. Gerrish, *President of Maine Central Institute.*—"Many years acquaintance with Greenleaf's Arithmetics, with a good opportunity to compare them with others, has led me to the conclusion that they are *the best extant*, and that their great popularity has been won by real merit."

From E. D. Sanborn, *Professor in Dartmouth College.*—"Greenleaf's works have all a deserved popularity, and their general use attests their excellence. *The most experienced teachers adopt them.*"

From W. L. P. Boardman, *now Principal of Lewis Grammar School, Boston.*—"I have always liked Greenleaf's Series of Arithmetics. The improvements contained in the new practical Arithmetic, make it in my estimation, one of the *very best* books before the public."

From I. Van Houten, *President of Board of Education, Paterson, N. J.*—"At the last meeting of the Board of Education of this city, Greenleaf's New Series was adopted for exclusive use in our schools."

From Alanson Palmer, *Superintendent of Schools, Long Island City, N. Y.*—"A trial of three years has satisfied all our principals of the general superiority of Greenleaf's Arithmetics, and all are favorable to the books. The New Practical we consider *perfection*, both as to arrangement and on account of new and important matter contained in it."

From George C. Fenn, *Principal of Grammar School, Ware, Mass.*—"For accuracy and conciseness, Greenleaf's New Series of Arithmetics, now used in all our schools, is unsurpassed."

From A. M. Beecher, *Principal of Seminary, Hartford, Ct.*—"We have recently adopted Greenleaf's Series. Our professor gives the books cordial commendation."

GREENLEAF'S NEW MATHEMATICAL SERIES.

From Asa Morehouse, *School Commissioner, Second District, Orange County, N. Y.*—"Greenleaf's New Series of Arithmetics are quite extensively used in this county, and wherever used they are giving good satisfaction."

From A. C. Brackett, *Principal of Storer Normal School, W. Va.*—"We shall hereafter use Greenleaf's Arithmetics, and no others, in this institution."

From D. C. Stone, *Professor in Oakland College, Cal.*—"I have used Greenleaf's Arithmetics for several years, and am not disposed to change."

From J. S. Cilley, *Principal of Graded School, Brandon, Vt.* — "We have used Greenleaf's Arithmetics in the Graded School for five years, and we shall continue to use them in years to come."

From Isaiah N. Leigh, *County Examiner, Hunterdon Co., N. J.*—"I have tested in the class-room, during a period of thirteen years, the different series of Arithmetics claimed to be the best, and found none to suit the wants of pupils better than Greenleaf's Series. Greenleaf's Arithmetics are used with great uniformity in this county.

From J. H. Walker, *County Examiner, Bergen County, Englewood, N. J.*—"Having used Greenleaf's Mathematical Text-books in the school-room for several years, it gives me pleasure to state that I *know of no books so well adapted* to the wants of our public schools."

From Dr. John M. Brewster, *Superintendent of Public Schools, Pittsfield, Mass.*—"Greenleaf's Arithmetics were introduced last Fall, by the unanimous vote of the School Committee, into all the public schools of Pittsfield; and all concerned, teachers and pupils, are glad of the change. We feel confident that *now* we are in possession of the best Arithmetics our pupils have ever used."

From Charles T. Whitman, *Town Supervisor of Schools, Norway, Me.*—"Greenleaf's New Practical Arithmetic *is the best Arithmetic we have ever used.* These and similar words of commendation are the universal sentiments of our teachers."

GREENLEAF'S
HIGHER SCHOOL MATHEMATICS.

NEW ELEMENTARY ALGEBRA.

This is a well-known popular favorite, and admitted on all hands to be a work of rare excellence. Its extraordinary success has led to the publication of a number of imitation treatises, but all falling far short of the merit of the original.

NEW SHORTER COURSE IN GEOMETRY.

The increased attention given of late to the application of Geometry in Elementary Education, has made a necessity a brief course like this. It contains all the more important theories and problems of the science, with exercises for original thought and practical applications. The success of this work shows that it is meeting a general want.

RECOMMENDATIONS.

From REV. GEORGE GANNETT, *Principal of the Gannett Institute for Young Ladies, Chester Square, Boston.*—"We have for some years, and exclusively, used in our Institute, Greenleaf's Mathematical Series. With these works, improved and perfected as they have been from time to time, I have been perfectly satisfied, and have no desire to exchange them for any others with which I am acquainted. They combine all the best qualities which I can conceive of as desirable in such books. With the New Elementary Algebra I am especially pleased."

From EDWARD G. WARD, *Principal of Public School No.* 11, *Jersey City, N. J.*—"My experience with Greenleaf's Mathematics justifies me in speaking of them in terms of the highest commendation. The Geometry, the Algebra, the Arithmetic,—are all good."

From J. D. SMITH, *Principal of Worcester Academy, Mass.*—"My experience has convinced me that Greenleaf's Elementary Algebra stands among *the very best* works on the subject. We use the work in this institution."

From GEORGE A. WENTWORTH, *Professor of Mathematics, Phillips Academy, Exeter, N.H.*—"I have used Greenleaf's New Elementary Algebra for more than six years. It has proved so satisfactory that I have no desire or inclination to change for any other."

From ISAAC N. CARLETON, *Principal of Normal School, Conn.*—"Although sorely pressed to adopt some other Algebra, we still continue to use Greenleaf's in the Normal School, and consider it *the best.*

GREENLEAF'S UNIVERSITY MATHEMATICS.

New Higher Algebra.

This elegant treatise, containing many important improvements, meets very fully the requirements of the best standard of instruction in Agricultural Colleges, Institutes of Technology and Universities.

Elements of Geometry.

Its arrangement is the simple order adopted by LEGENDRE. The acknowledged improvements of M. DECUNHA have been carefully incorporated into the work. Numerous miscellaneous Geometrical Exercises are a distinctive feature, and make the course both complete and practical.

Elements of Trigonometry.

In this comparatively short course, Plane and Spherical Trigonometry is *analytically* treated and *practically* applied. The Trigonometric Functions, in accordance with the modern method, have been regarded as *ratios.* The work is *full* and *comprehensive*, containing the most useful applications, with full *tables.*

IMPORTANT TESTIMONY.

From C. O. THOMPSON, *now Principal of Institute of Technical Science, Worcester, Mass.*—"Greenleaf's Algebras work well in my school. The New Elementary is logical, lucid, progressive, and dignified, and the NEW HIGHER is, if possible, better than the Elementary."

From CHARLES H. SMITH, *Professor of Mathematics, Bowdoin College, Me.*—"I have carefully examined Greenleaf's Algebras, Geometry, and Trigonometry of the New Comprehensive Series, and find they are admirably adapted to the purpose for which they are designed. Clear, sufficiently concise, and printed in excellent taste, they can hardly be surpassed as introductory text-books. Greenleaf's New Higher Algebra has long been a popular text-book in this college."

From J. D. RUNKLE, *now President of Massachusetts Institute of Technology, Boston.*—"The greatest care has evidently been taken with the style of the Geometry. After a careful search we have not been able to find one expression which we felt sure could be improved."

From SELDEN J. COFFIN, *Adjunct Professor of Mathematics, Lafayette College, Pa.*—"Greenleaf's New Higher Algebra, by its judicious arrangement and the practical nature of its examples, commends itself to the favor of teachers. The work excels in the mode of presenting the general theory of Equations; and the sections on Variation, Useful Formulas, and the chapter on Radical Quantities are of great practical value."

From E. J. THOMPSON, *Professor of Mathematics, University of Minnesota.*—"I have examined Greenleaf's Trigonometry, and like it well. I am pleased with its systematic and logical treatment of the subject, and also with the practical work it introduces to the student. I shall use it in my future classes."

PARKER'S EXERCISES IN COMPOSITION:

REVISED AND ENLARGED,

BY JAMES H. HAMILTON, M. D.

A COMPLETE COURSE IN ONE BOOK.

The attention of Teachers is specially invited to the improvements contained in the present edition of this useful MANUAL, which has been a great favorite in the School Room for so many years. It now appears in a new and more attractive dress, and in a form to render it more useful.

It is believed that "PARKER'S EXERCISES," in its revised and enlarged form, will prove a useful auxiliary in teaching Composition and Rhetoric, for the following reasons :—

1. The work is *progressive*, commencing with very easy and simple exercises, and concluding with the general principles of Rhetoric,—thus comprising a complete course in one book.

2. Capitals and Punctuation are clearly and fully treated where they should be, in the body of the work, and not in a brief appendix.

3. The whole work is divided into four distinct parts, presenting in order the different departments of Composition ; as words, sentences, discourse, etc.

4. A Chapter of Miscellaneous Exercises is given as a general Review of all the Lessons.

5. The exercises in this volume have stood the test of the School Room. Most of the original exercises have been retained, and a number of new ones added to render the work more complete.

This work has been approved for, and is now used in some of the most important schools of New York City, Boston, Worcester, Lowell, Providence, New Haven, etc., and in the public or private schools of many other important places in different States.

WHAT IS SAID OF IT.

From REV. EDWARD COOKE, D. D., *late Principal of Wesleyan Academy, Wilbraham, Mass.*—"We have used Parker's Progressive Exercises in English Composition, in the institution the past year and are well pleased with the work. It possesses the advantage of presenting just what the pupil needs to learn, and in the briefest possible manner. I know of no similar work which answers our wants so well; "*multum in parvo*," is practically realized in it."

From PROF. F. A. ALLEN, *late Principal of State Normal School, Pa.*—"I am glad to see this new little gem of a book, Parker's Progressive Exercises in Composition. Many thanks to Mr. Hamilton for revising this best of old standard works. We like the book and its division into four parts. Success to it.

From PROF. ALEXANDER WINCHELL, LL.D., *of University of Michigan.*—"I always liked Parker's Exercises in English Composition, and I like still more the revised and enlarged edition. My tastes, no less than occasional occupations, have led me to become acquainted with works of this class, and I do not express a recently formed opinion, when I say that Parker's Exercises has always had my highest approval. The writing of compositions becomes here an amusement. It is generally the dreaded drudgery of the school.

"*This work begins at the level of the lowest capacity, and floats him along without his consciousness, certainly without a wearisome effort, to the mastery of a finished style.* I presume it was intended for preparatory schools; but I am sure a Freshman in college could have no better rhetorical drill than to begin with the first lesson and faithfully follow on to the last. Small as the book is, unpretending as it is, it teaches exactly those things in which my experience informs me that even college students are very apt to be sadly deficient."

From JOSEPH HALL, A. M., *Principal of Public High School, Hartford, Ct.*—"We continue to use the revised edition of Parker's Exercises in English Composition, with constant and uniform success. I consider it to be by far the best work of the kind that I have ever seen."

From J. P. SHERMAN, A. M., *President Pennsylvania Female College.*—"I have used Parker's Exercises in English Composition, and have found it to be an excellent work, and one in which the young take great interest.

"The new enlarged edition is much improved, and will doubtless have a large circulation."

From H. DAME, *Principal of High School, Bangor, Me.*—"It gives me pleasure to state that Parker's Exercises in Composition has been in use in our High School during the past three years, to our entire satisfaction. I know of no work so admirably adapted to give the younger classes in a High School that training in the Art of English Composition, so essential to a good education, and yet, unfortunately, so often neglected, or crowded out by other studies. The principles are stated with clearness and conciseness, the exercises for practice are numerous and varied, and cover sufficient ground to occupy pupils, from two to three years; indeed, with proper teaching, it may be made to take the place of a regular text-book in Rhetoric. I can fully endorse it as a school-book leaving nothing to be desired."

PARKER'S EXERCISES IN COMPOSITION.

From E. O. Hovey, A. M., *Principal of High School, Newark, N. J.*—"Many thanks for Parker's Exercises in English Composition, revised by Hamilton. Parker's Exercises was always a good book, but it is now *the* English Composition *par excellence.*"

From Edwin Davenport, A. M., *St. John's School, Sing Sing, N. Y.*—"I have had occasion to admire in Parker's Exercises in English Composition, as revised by Dr. Hamilton, the completeness and the system of the work. Full as it certainly is, in every matter pertaining to the important Art of Composition, it is as far as possible from being an undigested mass of grammatical, rhetorical, and miscellaneous information. By a careful method the student is led to a practical acquaintance with all the requirements of correct expression even down to those last minutiæ of advice, which many a practical writer would find of no small service. *The system of the book, in its arrangement, divisions, judicious fullness of illustration, and examples for practice, leaves nothing to be desired.*

"The topic of Punctuation is treated in a particularly sensible way.

"The work is excellently adapted to pupils passing through the last three or four years of our schools. No other is so comprehensive and thorough."

THE QUESTION OF EXPENSE.

How great is the cost of time to the learner by the use of badly constructed text-books, or such as are behind the time. Since time is money, it may often be positive economy to adopt improved text-books. Labor-saving tools are as much a necessity in the school-room as elsewhere. A cost of a change at introduction rates, is hardly more than nominal.

The old book in use, when worn out would have to be replaced at full retail price. The introduced book can be had for first supply at two-thirds of the retail price, and if the one in use is surrendered, at half-price.

The question, then, of expense is readily answered. Books are the tools of the school-room, and you cannot afford to be without the best.

School Boards and Teachers,

are invited to correspond freely with us.

All of our educational publications are of a standard character, and will be supplied for *first introduction* at two-thirds retail prices. Or, when a corresponding old book in use is exchanged, at one-half of the retail price.

Address

ROBERT S. DAVIS & CO., Publishers, BOSTON,

OR THEIR INTRODUCING AGENTS:

ORLANDO LEACH,
142 and 144 Grand Street, New York.

BENJAMIN H. SANBORN,
Office at Brown & Gross' Book-store, Hartford,

Who will supply Committees and Teachers on Favorable Terms.

GREENLEAF'S

NEW PRIMARY ARITHMETIC,

FOR PRIMARY CLASSES.

104 pp. Price 25 Cents.

This new work excels all other First Lessons in numbers:

1. In the interesting and attractive nature of the *exercises.*

2. In the judicious use of *pictorial illustrations.*

3. In the practical methods of teaching the *tables* of addition subtraction, &c.

4. In the simplicity of the *inductive processes.*

5. In the number and beauty of its *pictorial illustrations,* which especially adapt it to the popular *Object Method* of instruction.

6. In containing a sufficient number of easy *Slate Exercises* for beginners.

7. In ordering call for *Greenleaf's New Primary Arithmetic.*

GREENLEAF'S

NEW ELEMENTARY ARITHMETIC,

AN EASY, COMPREHENSIVE COURSE OF

MENTAL AND WRITTEN EXERCISES,

Upon the Productive Plan, for Beginners.

208 pp. 16mo. Price 45 cents.

☞ This *new* work has been prepared expressly to provide for new educational demands. It excels:—

1. In being thoroughly *inductive*, *analytic*, and *practical*.

2. In combining *mental* and *written* exercises, and in applying the same forms of analysis to both.

3. In comprehending the essentials of arithmetic,—the *Fundamental Rules*, *Fractions*, *Analysis*, *Percentage*, and *Simple Interest*, *Metric System of Weights and Measures, etc.*

4. In especial adaptation to the want of pupils whose limited opportunities require that they should learn much in the shortest possible time.

☞ *This book is complete in itself, and, being cheap and comprehensive, is, pre-eminently,*

A BOOK FOR THE TIMES.

GREENLEAF'S

New Practical Arithmetic;

AN ENTIRELY NEW WORK,

FOR SCHOOLS AND SEMINARIES.

336 pp. Price 94 cents.

Especial attention is invited to its many

DISTINGUISHING CHARACTERISTICS.

1. ***The prominence given to the* ENUNCIATION *of* GENERAL PRINCIPLES, (pp. 13, 18, 26, 35, 44, etc.)**

2. ***The simple and clear treatment of* NOTATION *and* NUMERATION.**

3. ***The early introduction of the* DECIMAL POINT, (pp. 15, 16,) *and the explanation of the values expressed by figures at the right of the point*, (p. 52.)**

4. **EXCHANGE *of* COMMODITIES, BILLS AND INVOICES, ACCOUNTS, LEDGER COLUMNS, (pp. 72, 78,) *and various useful applications*, (pp. 142, 178, 186, 187, 202, 263, 212, etc.)**

5. ***The discussion* of FACTORING before Multiplication by Factors, (p. 83,) or Division by Factors, (p. 84.)**

6. ***The simplification* of DIVISION of *Common Fractions*, (p. 117,) *and of* DIVISION *of Decimals*, (p. 139.)**

7. **The introduction of COMPOUND DENOMINATE NUMBERS (p. 162,) *after Fractions, so as to avoid at least eight special rules in Reduction, Addition, and Subtraction of Denominate Numbers.***

8. ***The* METRIC SYSTEM *of* WEIGHTS *and* MEASURES *is in its proper place, in the body of the work*, (p. 155,) *and references made to the same in subsequent parts of the book*, (pp. 164, 166, 170, 187, 247, 283, etc.,) *as urged by* PROF. NEWTON, *of Yale College*, PRESIDENT HILL, *of Harvard University*, PROF. HENRY, *of Smithsonian Institute*, PROF. ROGERS, *of University of Pennsylvania*, CHANCELLOR CHAUVENET, *of Washington University*, ST. LOUIS, *and others.***

9. *The different cases or problems in* Simple Interest *are treated in orderly connection* (pp. 205, 211, 212.)

10. *The improved method of treating* Medial Proportion, (p. 298,) and the definite illustration of Evolution, (pp. 276, 277.)

11. *The presentation of important new topics and applications, as the* Cental (p. 153,) Government Interest, (pp. 205, 210,); Examples of Interest at $7\frac{3}{10}$ *per cent.* (pp. 206, 216,); Annual Interest, (p. 219,); New Vermont Interest Rule, Internal Revenue, (p. 252,); English Decimal Money Reckoning, (p. 258,) *etc*

12. *The avoiding of all that is* Obsolete, *or* Extraneous, *as Beer Measure,* the *Barley Corn, the Nail, Old Currencies, Duodecimals, Permutations, Combinations, etc.*

13. *The* Analytical Processes *throughout, making written Arithmetic in all its steps intellectual, never before to the same extent attained in any work.*

14. *The thorough enforcement of educational results, by means of systematic* Review Questions *and* Exercises, (pp. 32, 54, 58, 122, 186, etc.)

TIME SAVING AND LABOR SAVING.

It will be noted that this work with Greenleaf's New Primary, and New Elementary, or New Intellectual Arithmetics, forms a clear, practical, comprehensive system,

COMPLETE IN ONLY THREE BOOKS,

and is believed to be, the *cheapest, most convenient* and *productive*, that has ever appeared.

By its use, in comparison with certain *crude, overloaded,* and *long spun out* courses, of five or more books, with such catch penny accompaniments, as *Table Books, Dictation Cards, Books of Principles, Keys of Answers, Grammar School Editions,* etc., it is safe to say that the learner may not only *make better progress,* but, at least,

SAVE TWO YEARS OF VALUABLE TIME.